Solution to an exercise posed on Monday 17.11.2014

Theorem: (Lang's theorem for GL_n)

Let $K = \overline{\mathbb{F}_p}$, $G = GL_n$ over K. Consider

$$\sigma : GL_n \longrightarrow GL_n \quad , \quad g = (g_{ij}) \longmapsto \sigma(g) = (g_{ij}^{p}).$$

$$L : GL_n \longrightarrow GL_n \quad , \quad g \longmapsto L(g) = \sigma(g) \cdot g^{-1}.$$

Then L is surjective.

Claim 1: $d_e \sigma \equiv 0$ and $d_e L = -id$

Proof of claim 1:

On coordinate rings we have

$$\sigma^* : K[GL_n] \longrightarrow K[GL_n] \quad , \quad \sigma^*(x_{ij}) = x_{ij}^{p} \quad \forall i,j.$$

Then $\forall \delta \in T_e GL_n$:

$$d_e \sigma(\delta)(x_{ij}) = \delta \circ \sigma^*(x_{ij}) = \delta(x_{ij}^{p}) = p\, \delta(x_{ij}^{p-1}) \overset{\substack{\mathrm{char}\,K = p! \\ \downarrow}}{=} 0$$

$\Rightarrow d_e \sigma \equiv 0.$

For the second morphism consider:

$$L : GL_n \xrightarrow{\mathrm{diag}} GL_n \times GL_n \xrightarrow{(\sigma,\, \mathrm{inv})} GL_n \times GL_n \xrightarrow{m} GL_n.$$

Then:

$$d_e L(\delta) = d_{e,e} m \circ d_{e,e}(\sigma, \mathrm{inv})(\delta, \delta) =$$
$$= d_{e,e} m(d_e \sigma(\delta), -\delta) = d_e \sigma(\delta) - \delta \overset{d_e \sigma \equiv 0}{=} -\delta.$$

$\Rightarrow d_e L = -id.$

Claim 2: $\forall x \in GL_n$: $d_x L$ is bijective.

Proof of claim 2:

Consider the composition:

$$f : GL_n \xrightarrow{\cdot x} GL_n \xrightarrow{L} GL_n$$

$$g \longmapsto gx \longmapsto \sigma(gx) \cdot (gx)^{-1} = \sigma(g)\sigma(x) x^{-1} \cdot g^{-1}.$$

As above we may factor f as

$$f : GL_n \xrightarrow{\mathrm{diag}} GL_n \times GL_n \xrightarrow{(\sigma,\, \mathrm{inv})} GL_n \times GL_n \xrightarrow{(\cdot \sigma(x) x^{-1},\, id)} GL_n \times GL_n \xrightarrow{m} GL_n$$

$$\Rightarrow d_e f(\delta) = d_{\sigma(x)x^{-1},e}\, m\,(d_e(\cdot \sigma(x) x^{-1}) \circ \underset{\substack{\shortparallel \\ 0}}{d_e \sigma(\delta)}, -\delta) = d_{\sigma(x)x^{-1},e}(0, -\delta) = -\delta.$$

Thus $d_e f$ is bijective. Moreover $d_e(\cdot x): T_e GL_n \to T_x GL_n$ is bijective (with inverse $d_e(\cdot x^{-1})$).

$\Longrightarrow$ $d_e f = d_x L \circ d_e(\cdot x)$ implies that $d_x L$ bijective.

Claim 3: $L(GL_n)$ contains an open dense subset of GL_n.

Proof of claim 3:

Let $X = \overline{L(GL_n)}$. By an Alg-Geo-fact one can find an $x \in X$ s.th.

$$\dim X = \dim T_{L(x)} X$$

~~As $d_x L: T_x GL_n \to T_{L(x)} GL_n$ factors over X~~

As $L: GL_n \to GL_n$ factors over X, its derivation

$$d_x L : T_x GL_n \longrightarrow T_{L(x)} GL_n$$

factors over $T_{L(x)} X$. But by claim 2, $d_x L$ is bijective. Hence:

$$\dim T_{L(x)} X \geq \dim T_x GL_n = \dim GL_n.$$

$$\Longrightarrow \dim X \geq \dim GL_n$$

As X is a closed subvariety and GL_n irreducible, this implies

$$\overline{L(GL_n)} = X = GL_n$$

By another Alg-Geo fact (cf. [Springer, Linear algebraic groups, Theorem 1.9.5] this implies that $L(GL_n)$ contains an open dense subset of GL_n.

Claim 4: L is surjective.

Fix $h \in GL_n$ and consider

$$L_h : GL_n \to GL_n, \quad g \mapsto \sigma(g) \cdot h \cdot g^{-1}.$$

The computation done in claim 2 (for f) shows that $\forall x \in GL_n$ $d_x L_h$ is bijective as well. As we needed only bijectivity of $d_x L_h$ in claim 3, we see that $L_h(GL_n)$ contains some open subset as well.

$\Rightarrow L(GL_n) \cap L_h(GL_n)$ is non-empty

$\Rightarrow$ pick $y \in L(GL_n) \cap L_h(GL_n)$ and write

$$y = \sigma(g_1) \cdot g_1^{-1} = \sigma(g_2) \cdot h \cdot g_2^{-1}$$

for suitable $g_1, g_2 \in GL_n$

$\Rightarrow h = \sigma(g_2)^{-1} \sigma(g_1) \cdot g_1^{-1} \cdot g_2 = L(g_2^{-1} \cdot g_1)$

$\Rightarrow L$ surjective.