Even more problems discussed during exercise classes

P13: Let D_n be the symmetry group of a regular n-gon. Write D_n as a (non-trivial) semi-direct product.

P14:

Consider the group $G = \mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ given by the conjugation action

$$\varrho : \mathbb{Z}/3\mathbb{Z} \longrightarrow \operatorname{Aut}(\mathbb{Z}/7\mathbb{Z})$$
$$x \longmapsto \underbrace{2^x \cdot (-)}_{\text{mult. in } \mathbb{Z}}$$

i.e. $\forall a \in \mathbb{Z}/7\mathbb{Z}, b \in \mathbb{Z}/3\mathbb{Z}$: $\underbrace{b \cdot a \cdot b^{-1}}_{\text{mult. in } G} := \varrho(b)(a)$.

i) Compute $(1,2) \cdot (3,1)$ and $(3,1) \cdot (1,2)$ in G.

ii) Compute the inverse of $(2,1)$

iii) What are the 7-Sylow- and the 3-Sylow-subgroups of G?

P15: What is the radical $R(G)$ and the unipotent radical $R_u(G)$ for $G = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$?

P16: Lie-Kolchin revisited:

Consider two actions of the group T_3 :

i) on $V = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \end{pmatrix} \in K^{2,3} \right\}$ given by

$$g(A) = A \cdot g \quad \forall g \in T_3, A \in V$$

ii) on $W = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \subseteq K^{3,3} \right\}$ given by

$$g(A) = g^{-1} \cdot A \cdot g \quad \forall g \in T_3, A \in W.$$

Find in each situation a T_3-invariant flag in the respective vector space.