Some problems discussed during the exercise classes

P1: Describe all orbits of the action of

a) GL_n on K^n

b) $U_2 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ on K^2

c) $U_n = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ on K^n.

P2: Compute the dimension

$\dim D_n = n$; $\dim U_n = \frac{1}{2} n(n-1)$, ; $\dim T_n = \frac{1}{2} n(n+1)$

$\left(D_n = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}\right)$ $\left(U_n = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right)$ $\left(T_n = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}\right)$.

View now $D_n(\mathbb{R})$, $U_n(\mathbb{R})$ and $T_n(\mathbb{R})$ as real manifolds (with analytic topology) and compare their (analytic) dimension with the results above.

P3:

a) Let $X = K$, x the point 0. Compute $T(X, 0)$ via

- point derivations
- the dual of m_x / m_x^2.

b) Let $Y = V(xz - y) \subseteq K^3$. Compute $T(Y, (0,0,0))$.

c) Let $Z = V(x^2 - y^3) \subseteq K^2$. Compute $T(Z, (0,0))$.

P4:

Let $f: Y \to K^2$, $(x,y,z) \mapsto (x,y)$ (cf. E5).

Compute $d_{(0,0,0)} f : T(Y, (0,0,0)) \longrightarrow T(K^2, (0,0))$.

P5:

Let $f: X \to Y$ be a morphism s.th. its image consists of only one point. Show that $\forall x \in X$

$d_x f : T(X, x) \longrightarrow T(Y, f(x))$

is the zero morphism.

P6: Consider $\mathbb{P}^2 = \{(x_0 : x_1 : x_2)\}/_\sim$.

a) Describe all points of the subvariety $V(x_0 x_1 - x_2^2) \subseteq \mathbb{P}^2$.

b) Why does $V(x_0 x_1 - x_2) \subseteq \mathbb{P}^2$ ~~doesn't~~ not, make any sense?

P7: Consider the action $\mathbb{G}_m \curvearrowright \mathbb{P}^1$ given by

$$\lambda \cdot (x_0 : x_1) = (\lambda x_0 : x_1)$$

What are the orbits?

P8: Let char $K = p$ and consider

$$G = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a^p & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a \in K^\times,\ b \in K \right\} \subseteq GL_3 .$$

a) Check that G is a group.

b) Compute $\mathcal{L}(G)$.

c) Show that $[x, y] = 0 \;\; \forall x, y \in \mathcal{L}(G)$ while G is not commutative.

d) Show, $[\mathcal{L}(G), \mathcal{L}(G)] \neq \mathcal{L}((G, G)) = \mathcal{L}(\mathcal{D}(G))$.

e) Compute $Ad : G \longrightarrow GL(\mathcal{L}(G))$.

f) Show that $\ker Ad \neq Z(G)$ is not the center of G.