Solutions Sheet 1

E1, a) $X_1 = V(xy, x(x-1))$.

b) Closed subspaces of X_2 are given by ideals $I \subseteq K[x,y]$ containing $(x-y^2)$. But we have

$$\left\{\begin{array}{l}\text{Ideals } I \subseteq K[x,y] \\ \text{with } (x-y^2) \subseteq I\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{Ideals} \\ I \subseteq K[x,y]/(x-y^2)\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l}\text{Ideals} \\ I \subseteq K[y]\end{array}\right\}$$

$$(f(y), x-y^2) \longleftarrow\!\!\shortmid\ I = (f(y))$$

So any closed subset of X_2 has the form $V(f(y), x-y^2)$ for some polynomial f. If $\{y_1, \dots, y_n\}$ are the zeros of f, then $V(f(y), x-y^2) = \{(y_i^2, y_i), i = 1, \dots, n\}$ is a finite set (except for $f \equiv 0$, where $V(0, x^2-y^2) = X_2$).

Remark: Alternatively prove that $X_2 \cong \mathbb{K}$.

c) Write $f: K \to X_3$, $x \mapsto (p(x), q(x))$ for some polynomials $p, q \in K[x]$. Having an image in X_3 translates into the condition: $p(x) \cdot q(x) = 1 \quad \forall x \in K$.

$\Rightarrow$ Both polynomials are constant $p(x) \equiv a$, $q(x) \equiv a^{-1}$ for some $a \in K^\times$.

$\Rightarrow$ $f: K \to X_3$, $x \mapsto (a, a^{-1})$ has indeed only one point in its image.

d) We claim that $Z_1 = \{(0,a), a \in K\}$ and $Z_1' = \{(1,0)\}$ are the irred. components of X_1:

Z_1 is irreducible, because it is isomorphic to $\mathbb{K}$. Z_2 is irred., by definition, because it is just one point.

If $Y \subseteq X_1$ is another irred. component, then $Y \nsubseteq Z_1$ and $Y \nsubseteq Z_1'$ by maximality of irred. components. But then

$Y = (Y \cap Z_1) \cup (Y \cap Z_1')$

writes Y as a finite union of closed subspaces, contradicting irreducibility.

In the case X_2, we show that X_2 itself is its only irred. component. For this it suffices to see, that X_2 is irreducible. But this is immediate from the description of all closed subspaces given in part b).

E2) a) View $GL_2 = V((ad-bc)\cdot\Delta - 1) \subseteq K^5$ (where Δ corresponds to $\det^{-1}$). Let $h = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$ (corresponding to the point $(s,t,u,v,(sv-ut)^{-1})$). Then calculating the matrix ghg^{-1} explicitly, shows that c_h is given by

$$\begin{pmatrix} a \\ b \\ c \\ d \\ \Delta \end{pmatrix} \longmapsto \begin{pmatrix} (ads - act + bdu - bcv)\cdot\Delta \\ (a^2t + abv - abs - b^2u)\cdot\Delta \\ (d^2u + cds - cdv - c^2t)\cdot\Delta \\ (adv - bdu + act - bcs)\cdot\Delta \\ (sv-ut)^{-1} \end{pmatrix}$$

As the formulas on the r.h.s. are polynomials in (a,b,c,d,Δ), c_h is a morphism of affine varieties.

b) c_h is (as morphism of affine varieties) continuous for the Zariski topology. Hence

$$C_{GL_2}(h) = c_h^{-1}(\{h\})$$

is closed as the preimage of a closed set.

c) By continuity of c_h, each set $c_h^{-1}(H)$ is closed. Then

$$N_{GL_2}(H) = \bigcap_{h\in H} c_h^{-1}(H)$$

is closed as the intersection of closed subspaces.

E3) An easy calculation shows that G_1 is commutative, while G_2 is not. Thus they are not isomorphic as alg. groups. For the second part, consider G_1, G_2 as closed subspaces of $GL_2 \subseteq K^5$ given by

$$G_1 = V(a-d,\ c,\ (ad-bc)\cdot\Delta - 1) = V(a-d,\ c,\ a^2\cdot\Delta - 1) \subseteq K^5$$

$$G_2 = V(a-1,\ c,\ (ad-bc)\cdot\Delta - 1) = V(a-1,\ c,\ d\Delta - 1) \subseteq K^5$$

Then consider

$$G_1 \longrightarrow G_2 \qquad\qquad G_2 \longrightarrow G_1$$

$$\begin{pmatrix} a \\ b \\ c \\ d \\ \Delta \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ b \\ 0 \\ a \\ a\Delta \end{pmatrix} \qquad \begin{pmatrix} a \\ b \\ c \\ d \\ \Delta \end{pmatrix} \longmapsto \begin{pmatrix} d \\ b \\ 0 \\ d \\ \Delta^2 \end{pmatrix}$$

Now it is easy (and left to the reader), that both maps are well-defined and inverse to each other.

E4) Choose any injection $\rho : G \longrightarrow K$ with some image $\{a_1, \ldots, a_n\} \subseteq K$. As G is finite $\rho(G)$ is automatically closed in K, hence an affine variety.

It remains to see that the multiplication map and the inverse are algebraic, i.e. given by polynomials. We do this only for the multiplication map - the inverse is similar.

In other words, our task is to construct a polynomial $p(x,y) \in K[x,y]$ s.th. $\forall\, i,j \in \{1, \ldots, n\}$

$$p(a_i, a_j) = \rho\left(\rho^{-1}(a_i) \cdot \rho^{-1}(a_j)\right) \in K.$$

Such problems can be solved by Newton interpolation:

$$p(x,y) = \sum_{i,j} \left(\prod_{\substack{\ell \neq i}} \frac{x - a_\ell}{a_i - a_\ell} \cdot \prod_{m \neq j} \frac{y - a_m}{a_j - a_m} \cdot \rho\left(\rho^{-1}(a_i) \cdot \rho^{-1}(a_j)\right) \right).$$

Alternative solution:

First note that the symmetric group S_n can be seen as the closed subgroup of GL_n given by all permutation matrices. For general G, embed G into ~~some~~ S_n. Then G is a subgroup of a closed subgroup of GL_n, which (by finiteness of G) is necessarily closed itself. This way we can view G as a closed subgroup of GL_n, hence as an algebraic group.