Solutions to sheet 10

E37)

(a) There is a bijection (better: isomorphism of abstract groups)

$$X(\mathbb{Z}/d\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}/d\mathbb{Z}, \mathbb{G}_m) \xrightarrow{\sim} \{a \in \mathbb{G}_m \mid a^d = 1\}$$

$$\chi \longmapsto \chi(1)$$

$$\chi_a : \mathbb{Z}/d\mathbb{Z} \to \mathbb{G}_m,\ n \mapsto a^n \quad \longleftarrow\!\mid \quad a$$

With our ~~assump~~ assumptions on char K, we have

$$\mu_d := \{a \in \mathbb{G}_m \mid a^d = 1\} \cong \mathbb{Z}/d\mathbb{Z}$$

(i.e. there exists a primitive d^{th} root of unity).

$$\Longrightarrow X(\mathbb{Z}/d\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$$

(b) The same approach as in a) yields now

$$X(\mathbb{Z}/p^n\mathbb{Z}) \cong \{a \in \mathbb{G}_m \mid a^{p^n} = 1\} \underset{\substack{\uparrow \\ \mathrm{char}\,K = p!}}{=} \{1\}$$

So $X(\mathbb{Z}/p^n\mathbb{Z})$ is the trivial group.

E38)

(a) Define:

$$Y(D_n) \times Y(D_n) \longrightarrow Y(D_n)$$

$$(\eta, \eta') \longmapsto \eta + \eta' : \mathbb{G}_m \longrightarrow D_n,\ g \longmapsto \eta(g) \underset{\substack{\uparrow \\ \text{group mult. in } D_n}}{\cdot} \eta'(g)$$

Associativity: $\forall g \in G$ and $\eta, \eta', \eta'' \in Y(D_n)$:

$$((\eta + \eta') + \eta'')(g) = (\eta(g) \cdot \eta'(g)) \cdot \eta''(g) =$$

$$= \eta(g) \cdot (\eta'(g) \cdot \eta''(g)) = (\eta + (\eta' + \eta''))(g).$$

Inverse:

Let $\iota : D_n \longrightarrow D_n$ be the inverse morphism for the group D_n. Then define for $\eta \in Y(D_n)$

$$(-\eta) := \iota \circ \eta \in Y(D_n)$$

Identity element: The morphism with constant image the identity in D_n.

(b) Choose any isomorphism $D_n \cong (\mathbb{G}_m)^n$. Then

$$Y(D_n) = Hom(\mathbb{G}_m, D_n) \cong Hom(\mathbb{G}_m, \mathbb{G}_m^n) \cong$$
$$\cong (Hom(\mathbb{G}_m, \mathbb{G}_m))^n = X(\mathbb{G}_m)^n \cong \mathbb{Z}^n.$$

More explicidely we can define

$$\mathbb{Z}^n \longrightarrow Y(D_n)$$
$$(\lambda_i)_{i=1,\dots,n} \longmapsto \eta_\lambda : \mathbb{G}_m \longrightarrow D_n$$
$$g \longmapsto (g^{\lambda_i})_{i=1,\dots,n}$$

diagonal matrix.

(c) ~~...~~ Fix the isomorphisms

$\mathbb{Z}^n \xrightarrow{\sim} Y(D_n)$ as in (b)

$\mathbb{Z}^n \xrightarrow{\sim} X(D_n)$, $(\lambda_i)_{i=1,\dots,n} \longmapsto \chi_\lambda : D_n \longrightarrow \mathbb{G}_m$

$$(g_i)_{i=1,\dots,n} \longmapsto \prod g_i^{\lambda_i}.$$

Using these identifications, we can compute the matrix associated to the pairing

$$\mathbb{Z}^n \times \mathbb{Z}^n \xrightarrow{\sim} X(D_n) \times Y(D_n) \xrightarrow{\langle , \rangle} \mathbb{Z}$$

Let $e_i = (0, 0, \dots, \underset{\substack{\uparrow \\ i\text{th position}}}{1}, 0, \dots, 0)$. Then $\langle e_i, e_j \rangle$ corresponds

to the morphism

$$\mathbb{G}_m \longrightarrow D_n \longrightarrow \mathbb{G}_m$$
$$g \longmapsto (1, \dots, 1, \underset{\substack{\uparrow \\ j\text{-th position}}}{g}, 1, \dots, 1) \quad (g_1, \dots, g_n) \longmapsto g_i$$

which is the identity morphism for $i = j$ and the trivial morphism mapping everything to $1 \in \mathbb{G}_m$ for $i \neq j$.

Thus the pairing $\langle , \rangle$ is given by $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ (~~for~~ wrt. the basis $\{e_i\}_i$).

Now consider $X(D_n) \longrightarrow Y(D_n)^*$. Let $\alpha \in Y(D_n)^*$. Then the element $\sum_i \alpha(e_i) \cdot e_i \in \mathbb{Z}^n \cong X(D_n)$ is a preimage. Thus $X(D_n) \longrightarrow Y(D_n)^*$ is a surjective morphism between free $\mathbb{Z}$-modules of the same rank. Hence it is an isomorphism.

The same goes for $Y(D_n) \longrightarrow X(D_n)^*$

E39)

a) Let $d \geq 2$ and $\eta' \in Y(D_n)$. Choose $\xi \neq 1$ a dth root of unity in K. Then by definition (for $\eta' = (\lambda_1, \dots, \lambda_n)$ under $Y(D_n) \cong \mathbb{Z}^n$):

$$d\eta'(\xi) = (\xi^{\lambda_i \cdot d})_i = (1)_i$$

$\Rightarrow \xi \in \ker(d\eta')$ and $d\eta'$ is not injective.

b) If there exists such a $\chi \in X(D_n)$, then

$\chi \circ \eta = id_{\mathbb{G}_m}$ is injective

$\Rightarrow \eta$ is injective.

Conversely assume η injective. Then part a) implies that $Y(D_n)/\mathbb{Z}\cdot\eta$ ~~exists~~ has no torsion elements. Hence it is isomorphic to $\mathbb{Z}^{n-1}$ ($rk\, Y(D_n) = n \Rightarrow rk(Y(D_n)/\mathbb{Z}\eta) = n-1$).

Fix now a $\mathbb{Z}$-basis $(v_1, \dots, v_{n-1})$ of $Y(D_n)/\mathbb{Z}\eta$, i.e. elements $v_1, \dots, v_{n-1}$ s.th. any other element in $Y(D_n)/\mathbb{Z}\eta$ can be written uniquely in the form $\sum_i \lambda_i v_i$ for some $\lambda_i \in \mathbb{Z}$.

Choose now any preimage $\tilde{v}_i$ of v_i in $Y(D_n)$. Then $(\eta, \tilde{v}_1, \dots, \tilde{v}_{n-1})$ is a $\mathbb{Z}$-basis of $Y(D_n)$ (copy the proof for vector spaces!).

$\Rightarrow$ We may define $\alpha \in Y(D_n)^*$ by

$$\alpha\left(\sum_{i=1}^{n-1} \lambda_i \tilde{v}_i + \lambda_n \eta\right) = \lambda_n$$

By E38 c) we may now find some $\chi \in X(D_n)$ with

$$\langle \chi, - \rangle = \alpha$$

and in particular:

$$\chi \circ \eta = \langle \chi, \eta \rangle = \alpha(\eta) = 1 \in \mathbb{Z} \cong Hom(\mathbb{G}_m, \mathbb{G}_m)$$

$\Rightarrow \chi \circ \eta = id_{\mathbb{G}_m}$ as desired.

E40)

a) For each $\chi \in X(G)$, $\ker \chi \subseteq G$ is a closed normal subgroup. Furthermore any intersection of closed normal subgroups is again closed and normal

$\Rightarrow H \leq G$ is closed and normal.

b) By the universal property of quotients, any $\chi \in X(G)$, $\chi : G \to \mathbb{G}_m$ factors through some $\bar{\chi} : G/H \to \mathbb{G}_m$.

Now pick any element $g \in G/H$, $g \neq 1$. By definition of H, there exist some $\bar{\chi} : G/H \to \mathbb{G}_m$ with $\bar{\chi}(g) \neq 1$.

$\Rightarrow \bar{\chi}(g)$ not unipotent

$\Rightarrow g$ not unipotent

$\Rightarrow 1 \in G/H$ ~~K~~ is the only unipotent ~~group~~ element

$\Rightarrow$ Alle elements in G/H are semisimple.

Moreover $\forall \chi \in X(G)$, $g, h \in G$:

$$\chi(ghg^{-1}h^{-1}) = \chi(g)\chi(h)\chi(g^{-1})\chi(h^{-1}) = $$
$$= \chi(g)\chi(g^{-1})^{-1}\chi(h)\chi(h)^{-1} = 1.$$

$\Rightarrow \forall \chi \in X(G) : \quad (G, G) \subseteq \ker \chi$

$\Rightarrow (G, G) \subseteq H$

$\Rightarrow G/H$ is a quotient of the abelian group $G/(G,G)$

$\Rightarrow G/H$ is abelian

But any abelian ~~semi simple~~ group with all elements semi-simple is diagonalizable.

c) We have mutually inverse morphisms

$$X(G) \rightleftarrows X(G/H)$$

$$\chi \longmapsto \bar{\chi} \quad \text{(see part b))}$$

$$G \to G/H \xrightarrow{\bar{\chi}} \mathbb{G}_m \longleftarrow \bar{\chi}$$

$\Rightarrow X(G) \cong X(G/H)$.

~~$\Rightarrow$~~ By part b) $X(G/H)$ is finitely generated, because G/H is diagonalizable. $\Rightarrow X(G)$ is finitely generated.