Solutions to Sheet 11

E41)

Claim 1: The $V[\chi]$ are linearly independent.

Proof: We proceed as in the proof of linear independence of characters (lemma 7.3). Choose $\chi_1, \dots, \chi_n \in X(G)$ with n minimal and $v_i \in V[\chi_i]$ $\forall i$ such that $\sum v_i = 0$. Choose $\tau \in G$ s.th. $\chi_1(\tau) \neq \chi_2(\tau)$. Then:

$$\sum_i \chi_i(\tau) \cdot v_i = \sum_i \rho(\tau)(v_i) = \rho(\tau)(\sum v_i) = \rho(\tau)(0) = 0.$$

$$\Rightarrow 0 = \chi_1(\tau) \cdot \sum_i v_i - \sum_i \chi_i(\tau) \cdot v_i = \sum_{i \geq 2} (\chi_1(\tau) - \chi_i(\tau)) v_i$$

As $\chi_1(\tau) - \chi_2(\tau) \neq 0$, we got a contradiction to our assumption n minimal.

Claim 2: The $V[\chi]$ span V.

Proof: G is a torus, hence diagonalizable. Thus we may choose a basis $v_1, \dots, v_m$ of V s.th. G acts via diagonal matrices wrt. this basis. Consider now for each $i = 1, \dots, m$ the character

$$\chi_i : G \xrightarrow{\rho} D_m \xrightarrow{pr_i} G_m$$
$$\operatorname{diag}(a_1, \dots, a_m) \longmapsto a_i$$

Then by choice of the basis $\{v_i\}_i$ we have $\forall g \in G$

$$\rho(g)(v_i) = \chi_i(g) \cdot v_i$$

$\Rightarrow v_i \in V[\chi_i]$

$\Rightarrow v_i$ span V.

E42)

(a) View the closed subgroup $G_m \subseteq D_n$ as a cocharacter $\eta : G_m \to D_n$.

By E39, choose $\chi : D_n \to G_m$ s.th. $\chi \circ \eta = id_{G_m}$ and define

$H' = \ker(\chi)$.

To see that $m : G_m \times H' \longrightarrow D_n$, $(h, h') \mapsto h \cdot h'$ is an isomorphism, we give the inverse as $m^{-1} : D_n \longrightarrow G_m \times H'$, $g \mapsto (\chi(g), g \cdot \chi(g^{-1}))$.

Indeed: $m \circ m^{-1}(g) = m(\chi(g), g\chi(g^{-1})) = \chi(g) \cdot g \cdot \chi(g)^{-1} = g$.

and $m^{-1} \circ m(h, h') = m^{-1}(h \cdot h') = (\chi(h h'), h \cdot h' \cdot \chi(h \cdot h')^{-1}) = (h, h \cdot h' \cdot h^{-1}) = (h, h')$.

(b) We proceed quite similarly: Choose any embedding $G \hookrightarrow D_n$. Then (after choosing an isomorphism $H \cong \mathbb{G}_m^d$ for some d) the composition $H \xrightarrow{\eta} G \hookrightarrow D_n$ is given by a d-tuple of cocharacters $(\eta_1, \dots, \eta_d)$. Choose by E39, for each η_i a character χ_i with $\chi_i \circ \eta_i = \mathrm{id}$. This defines a morphism

$$\chi = (\chi_1, \dots, \chi_d) : D_n \longrightarrow H$$

such that $\chi \circ \eta = \mathrm{id}_H$. We now define:

$$H' = G \cap \ker(\chi) = G \cap \bigcap_i \ker(\chi_i).$$

Via the same computations as in (a) the multiplication morphism $m : H \times H' \longrightarrow G$ has an inverse

$$m^{-1} : G \longrightarrow H \times H', \quad g \mapsto (\chi(g), g \cdot \chi(g)^{-1}).$$

E43)

(a) If g is semi-simple, then its conjugacy class is closed by theorem 8.3. If g is not semi-simple, assume wlog that it is given by its Jordan normal form, which has non-trivial entries off the diagonal. Then for all $a \in K$, $a \neq 0$, the matrix with ~~coordinates~~ coefficients a_{ij}

$$a_{ij} = \begin{cases} \text{same as } g & \text{if } i = j \\ a & \text{if } i \neq j \text{ and coeff. of } g \text{ is equal to } 1 \\ 0 & \text{otherwise} \end{cases}$$

lies in the conjugacy class of g. Hence the closure of the conjugacy class contains the diagonal matrix with diagonal entries as in g (but 0 otherwise), which does not lie in the conjugacy class of g. Hence this conjugacy class cannot be closed.

(b) Consider the unipotent groups G acting on itself by conjugation. Then the conjugacy class of some $g \in G$ is just the orbit of this action, which is automatically closed by E36).

E44)

(a) We first compute the centralizer in GL_5:

$$s \cdot \begin{pmatrix} a_{11} & \dots & a_{15} \\ \vdots & \ddots & \vdots \\ a_{51} & \dots & a_{51} \end{pmatrix} s^{-1} = \begin{pmatrix} a_{11} & \dots & a_{13} & -a_{14} & -a_{15} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{31} & \dots & a_{33} & -a_{34} & -a_{35} \\ -a_{41} & \dots & -a_{43} & a_{44} & a_{45} \\ -a_{51} & \dots & -a_{53} & a_{54} & a_{55} \end{pmatrix}$$

Hence $C_{GL_5}(s) = GL_3 \times GL_2 \subseteq GL_5$.

$\Rightarrow C_H(s) = H \cap (GL_3 \times GL_2) = O_3 \times O_2 \subseteq O_5$.

By E16) we have $\mathcal{L}(O_n) = \{A \in K^{n\times n} \mid A = -A^T\}$.

$\Rightarrow \mathcal{L}(C_H(s)) = \mathcal{L}(O_3 \times O_2) = \mathcal{L}(O_3) \oplus \mathcal{L}(O_2) =$

$= \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, A_1 \in K^{3\times 3}, A_2 \in K^{2\times 2}, A_1 = -A_1^T, A_2 = -A_2^T \right\}$.

(b) Again by E16) we have

$\mathcal{L}(H) = \mathcal{L}(O_5) = \{A \in K^{5\times 5} \mid A = -A^T\}$.

Then: $c_{\mathcal{L}(H)}(s) = \{A \in \mathcal{L}(H) \mid Ad(s)(H) = H\} =$

$= \{A \in \mathcal{L}(H) \mid sHs^{-1} = H\} =$ (same computation as above)

$= \mathcal{L}(H) \cap (K^{3\times 3} \oplus K^{2\times 2}) =$

$= \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, A_1 \in K^{3\times 3}, A_2 \in K^{2\times 2}, A_1 = -A_1^T, A_2 = -A_2^T \right\}$.