# Solutions to Sheet 12

E 45)

(a) A similar computation as in E 44(a) shows

$$C_{GL_3}(s) = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}. \quad \Longrightarrow \quad C_{U_3}(s) = \left\{\begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}.$$

(b) Let $g = \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}$. Then

$$\gamma_s(g) = sgs^{-1}g^{-1} = \begin{pmatrix} 1 & a & \lambda\mu^{-1}b \\ & 1 & \lambda\mu^{-1}c \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & -a & ac-b \\ & 1 & -c \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & (\lambda\mu^{-1}-1)\cdot b \\ & 1 & (\lambda\mu^{-1}-1)\cdot c \\ & & 1 \end{pmatrix}$$

Hence with $\lambda \neq \mu$ we get:

$$\gamma_s(U_3) = \begin{pmatrix} 1 & 0 & * \\ & 1 & * \\ & & 1 \end{pmatrix} \subseteq U_3.$$

(c) Injectivity:

~~Let $(g,h) \in \ker(m)$~~ Let $(g,h), (g',h') \in C_{U_3}(s) \times \gamma_s(U_3)$ with $m(g,h) = m(g',h')$. Then $g \cdot h = g' \cdot h' \Rightarrow g'^{-1} \cdot g = h' \cdot h^{-1}$ in $GL_3$. Moreover the explicit description of $\gamma_s(U_3)$ shows, that $\gamma_s(U_3)$ is a group.

$$\Longrightarrow g'^{-1} \cdot g = h' \cdot h^{-1} \in C_{U_3}(s) \cap \gamma_s(U_3) = \left\{\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}\right\}.$$

$\Longrightarrow g = g'$ and $h = h'$ showing injectivity.

Surjectivity:

Let $\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} \in U_3$. Then:

$$\underbrace{\begin{pmatrix} 1 & a & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}}_{\in\, C_{U_3}(s)} \cdot \underbrace{\begin{pmatrix} 1 & 0 & b-ac \\ & 1 & c \\ & & 1 \end{pmatrix}}_{\in\, \gamma_s(U_3)} = \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}.$$

E 46)

(a) Surjectivity is obvious. Let $g = \begin{pmatrix} 1 & \lambda \\ & 1 \end{pmatrix} \in U_2$, $x = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in T_2$.

Then $$f(g(x)) = f\left(\begin{pmatrix} 1 & \lambda \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ & c \end{pmatrix} \cdot \begin{pmatrix} 1 & -\lambda \\ & 1 \end{pmatrix}\right) =$$
$$= f\left(\begin{pmatrix} a & b+\lambda c-\lambda a \\ & c \end{pmatrix}\right) = \begin{pmatrix} a & \\ & c \end{pmatrix} = \cancel{f\begin{pmatrix} a & b \\ & c \end{pmatrix}}\ f(x)$$

$\Rightarrow f$ is ~~the~~ constant on $U_2$-orbits.

(b) We already computed $g(x)$ in part (a). This shows that $x = \begin{pmatrix} a & b \\ & c \end{pmatrix}$ is invariant under $U_2$ if and only if

$$b + \lambda c - \lambda a = b \quad \forall \lambda \in K$$

or equivalently if and only if $a = c$.

$$\Rightarrow C_{T_2}(U_2) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, a \in K\setminus\{0\}, b \in K \right\}.$$

Then $f(C_{T_2}(U_2)) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \in K\setminus\{0\} \right\} \subsetneqq D_2$ is not surjective.

E 47)

(a) Lemma:

Quotients of tori are again tori.

Proof:

A quotient of a torus is abelian and consists of semi-simple elements (as images of semi-simple elements). Hence such a quotient is diagonalizable. Moreover it is connected (as the image of a connected group), hence indeed a torus.

Now consider the torus $D_n \leq GL_n$ and let $T \leq PGL_n$ be its image under $GL_n \longrightarrow PGL_n$. As $Z(GL_n) \leq D_n$ we have an isomorphism $T \cong D_n / Z(GL_n)$. Thus $T$ is a torus by the Lemma. Moreover $\dim T = \dim D_n - \dim Z(GL_n) = n - 1$.

(b) For $g \in GL_n$ write $\bar{g} \in PGL_n$ for its image. Then:

$$\{ g \in GL_n \mid \bar{g} \in N_{PGL_n}(T) \} = \{ g \in GL_n \mid \bar{g} T \bar{g}^{-1} = T \} =$$
$$= \{ g \in GL_n \mid \bar{g} \bar{t} \bar{g}^{-1} \in T \quad \forall t \in D_n \} =$$
$$= \{ g \in GL_n \mid g t g^{-1} \in D_n \quad \forall t \in D_n \} = N_{GL_n}(D_n).$$

As $Z(GL_n) \leq N_{GL_n}(D_n)$ this implies:

$$N_{PGL_n}(T) \cong N_{GL_n}(D_n) / Z(GL_n).$$

$$\Rightarrow W(PGL_n) = N_{PGL_n}(T) / T \cong \left( N_{GL_n}(D_n) / Z(GL_n) \right) \Big/ \left( D_n / Z(GL_n) \right) \cong N_{GL_n}(D_n) / D_n =$$
$$= W(GL_n) = S_n \quad \text{(by example 7.9 from the lecture).}$$

E48,

(a) Consider $T_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, A \in SO_2 \right\} = \left\{ \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a^2+b^2 = 1 \right\}$

A direct computation shows that $T_0$ is abelian. Moreover $T_0$ is connected and consists only of semi-simple elements (as another direct computation shows). Thus $T_0$ is a torus.

One can even give an isomorphism explicitely:

$$\mathbb{G}_m \longrightarrow T_0$$
$$\lambda \longmapsto \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for } a = \frac{1}{2}(\lambda + \lambda^{-1}), \; b = \frac{1}{2i}(\lambda - \lambda^{-1}).$$

(where $i \in K$ with $i^2 = -1$).

(b) Claim:

Any $A \in SO_3$ has an eigenvalue equal to $1$ or $-1$.

Proof of claim:

$A^{-1} = A^T$ implies for char. polynomials $\chi(A^{-1}) = \chi(A^T) = \chi(A)$. Hence if $\lambda$ is an eigenvalue, then so is $\lambda^{-1}$. As there are three eigenvalues, one of them has to satisfy $\lambda = \lambda^{-1}$, i.e. is equal to $\pm 1$.

Consider now any 1-dim. torus $T \subseteq SO_3$. Then we may diagonalize $T$ inside $GL_3$. The corresponding morphism $\mathbb{G}_m \cong T \longrightarrow D_3 \subseteq GL_3$ is given by three cocharacters, say $\eta_1, \eta_2$ and $\eta_3$.

By the claim, there is for each $t \in T$ some $\eta_i$ with $\eta_i(t) \in \{\pm 1\}$.

As $\{t \in T \mid \eta_i(t) \in \{\pm 1\}\}$ is closed in $T$, irreducibility of $T$ implies, that there is one $\eta_i$ with $\eta_i(t) \in \{\pm 1\} \; \forall t \in T$. But then $\eta_i$ has to be constant and it follows $\eta_i \equiv 1$.

$\Rightarrow$ There exists a vector $v$ fixed by all elements of $T$.

Extending $v$ to an orthonormal basis, we see that we may conjugate $T$ by an element in $SO_3$ to some torus inside

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix} \cap SO_3 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, A \in SO_2 \right\} = T_0$$

As $T_0$ has no non-trivial proper subtorus, we see that $T$ is conjugate to $T_0$.

(c) Proof 1: Repeat the arguments of part (b), but with morphisms $\eta_i : \mathbb{G}_m^2 \to \mathbb{G}_m$ instead of cocharacters. This shows that ~~any~~ a 2-dim. torus is conjugate to a subtorus of $T_0$. As $\dim T_0 = 1$, we get a contradiction.

Proof 2: Let $T \subseteq SO_3$ be a 2-dim. torus. By part (b) assume wlog $T_0 \subseteq T$. Then an explicit calculation shows

$$T \subseteq C_{SO_3}(T_0) = T_0 \cup \left\{ \begin{pmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & -1 \end{pmatrix} \;\middle|\; a^2 + b^2 = 1 \right\}$$

and we see $\dim C_{SO_3}(T_0) = \dim T_0 = 1$ Contradiction.