Solutions to Sheet 13

E49)

a) Let $\tau \in S_n$ be any transposition. Then let $H = \langle \tau \rangle \cong \mathbb{Z}/2$. We claim $S_n \cong A_n \rtimes H$: Indeed, $A_n \subseteq S_n$ is normal, $A_n \cap H = \{e\}$ (because $\tau \notin A_n$) and $A_n \subsetneq A_n \cdot H \subseteq S_n$. Now $[S_n : A_n] = 2$ implies that $A_n \cdot H = S_n$ and we get the semi-direct product structure.

b) ~~Let $H = Z(GL_n) = \{\lambda \cdot 1_n, \lambda \in K^*\}$~~

Let $H = \{\operatorname{diag}(\lambda, 1, 1, \dots, 1),\ \lambda \in K^*\} \subseteq GL_n$. Then $SL_n \subseteq GL_n$ normal and $SL_n \cap H = \{e\}$ is obvious.
Moreover any $A \in GL_n$ can be written as

$$A = \underbrace{\operatorname{diag}(\det A, 1, \dots, 1)}_{\in H} \cdot \underbrace{\left(\operatorname{diag}(\det A^{-1}, 1, \dots, 1) \cdot A\right)}_{\in SL_n}$$

$\Rightarrow SL_n \cdot H = GL_n$

$\Rightarrow GL_n = SL_n \rtimes H$.

c) If $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2 \rtimes \mathbb{Z}/2$, then $\mathbb{Z}/4\mathbb{Z}$ must contain at least two different subgroups isomorphic to $\mathbb{Z}/2$. But ~~is~~ has only one of them, namely $2\mathbb{Z}/4\mathbb{Z}$. Contradiction.

E50)

a) Let $G = GL_n$ and $H \cong S_n$ the subgroup of all permutation matrices (and assume either $\operatorname{char} K = 0$ or $\operatorname{char} K > 0$ & $\operatorname{char} K \nmid n!$).
Then H is not commutative, hence cannot lie in some maximal torus. But H consists of semi-simple elements (due to our assumption on ~~the~~ $\operatorname{char} K$).

Remark:

If $\operatorname{char} K = p \mid n!$, then choose a prime $\ell \neq p$ and let $H \subseteq S_n$ be a ℓ-Sylow subgroup. If $n \gg \ell$ then H is again not abelian and we can argue as above.

b) Let ~~...~~ $G = GL_n$, $H = T = D_n$ max. torus. Then

$$C_G(H) = D_n$$

but $N_G(H) = \{\text{monomial matrices}\}$ (cf. exercise 8)

E 51)

G is an extension of two solvable groups, hence it is solvable itself. ~~By the structure~~

Lemma: G is connected

Proof: Assume we have connected components $Z_0 = G^\circ, Z_1, \dots, Z_n$. Consider their images under $\pi: G \longrightarrow G/T$. $\pi(Z_0)$ is closed as the image of a closed subgroup. Then $\pi(Z_i)$ is closed as well, because it is a translate of $\pi(Z_0)$ by some element in G. As π is surjective, the $\pi(Z_i)$ must cover G/T. Hence there exists some $x \in G/T$ s.th. x lies in the image of at least two of the $\pi(Z_i)$. Then $\pi^{-1}(x)$ cannot be connected. But all fibers are isomorphic to $T = \ker(\pi)$, which is connected. Contradiction.

By the structure theorem of connected solvable groups, we can write: $G \cong R_u(G) \rtimes T_0$ for some maximal torus T_0. Hence it suffices to show that $R_u(G) = G_u = \{e\}$. So assume $g \in G$ is a unipotent element. Then $\pi(g) \in G/T$ is again unipotent. Because G/T is a torus, we get $\pi(g) = e$, i.e. $g \in T = \ker(\pi)$. But T is another torus and thus $g = e$.

q.e.d.

E52)

$C_G(T) \subseteq G$ is a closed subgroup of a solvable subgroup, hence solvable itself. Moreover it was shown in the lecture, that $C_G(T) = N_G(T)$ is connected.

Now $T \subseteq G$ was maximal, so $T \subseteq C_G(T)$ is a maximal torus in $C_G(T)$. Hence by the structure theorem, we may write

$$C_G(T) \cong R_u(C_G(T)) \rtimes T$$

By definition $T \subseteq C_G(T)$ lies in the center, i.e. any element in $R_u(C_G(T))$ commutes with any element in T. So by the construction of semi-direct product, we ~~get~~ actually get a direct product decomposition

$$C_G(T) \cong R_u(C_G(T)) \times T$$

Now both T and $R_u(C_G(T))$ are nilpotent, hence so is their direct product.