Solutions to Sheet 14

E 53)

a) Let $\pi : G \longrightarrow G/N$ be the projection.

Then $\pi(R(G))$ is connected, because $R(G)$ is.

$\pi(R(G))$ is solvable as a quotient of a solvable group.

$\pi(R(G))$ normal as the image of a normal subgroup under a surjective morphism.

$\Rightarrow \pi(R(G)) \subseteq R(G/N)$.

Same arguments for unipotent instead of solvable gives

$\pi(R_u(G)) \subseteq R_u(G/N)$.

b) $(R(G) \cap N)^\circ$ is by definition connected. As a subgroup of the solvable group $R(G)$, it is solvable.

$R(G)$ is normal in G, so $R(G) \cap N$ is normal in N.

$\Rightarrow (R(G) \cap N)^\circ$ normal in N

$\Longrightarrow (R(G) \cap N)^\circ \subseteq R(N)$.

$R(N)$ is connected and solvable. We show that it is normal in G:

Let $\alpha : N \longrightarrow N$ be any automorphism. Then $\alpha(R(N))$ is again connected, solvable and normal in N. Hence by maximality of $R(N)$ we get: $\alpha(R(N)) = R(N)$.

Now for any $g \in G$, conjugation with g defines an automorphism of N. Hence the previous arguments imply

$g \cdot R(N) \cdot g^{-1} = R(N) \quad \forall g \in G$.

$\Rightarrow R(N)$ is normal in G

$\Longrightarrow R(N) \subseteq \cancel{(R(G) \cap N)}\ R(G)$

With $R(N) \subseteq N$, we get $R(N) \subseteq R(G) \cap N$.

With $R(N)$ connected, we get $R(N) \subseteq (R(G) \cap N)^\circ$.

Same arguments for $R_u(N) = (R_u(G) \cap N)^\circ$.

E 54)

Claim: $R_u(G) = \begin{pmatrix} \mathbb{1}_m & * \\ 0 & \mathbb{1}_n \end{pmatrix} \subseteq G$.

Proof: A direct computation shows that $\begin{pmatrix} \mathbb{1}_m & * \\ 0 & \mathbb{1}_n \end{pmatrix}$ is normal.

Then:

$$G / \begin{pmatrix} \mathbb{1}_m & * \\ 0 & \mathbb{1}_n \end{pmatrix} \cong SL_m \times GL_n$$

$$\left[\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}\right] \longmapsto (A, B)$$

Now $SL_m \times GL_n$ is reductive (as a direct product of two reductive groups). $\Rightarrow R_u(G) = \begin{pmatrix} \mathbb{1}_m & * \\ 0 & \mathbb{1}_n \end{pmatrix}$.

Claim: $R(G) = \left\{ \begin{pmatrix} \mathbb{1}_m & * \\ 0 & \lambda \mathbb{1}_n \end{pmatrix}, \lambda \in K^* \right\} \subseteq G$.

Proof:

This group is the kernel of

$$G \twoheadrightarrow SL_m \times GL_n \twoheadrightarrow SL_n \times PGL_n$$

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \longmapsto (A, B)$$

$$(A, B) \longmapsto (A, [B]).$$

and $SL_n \times PGL_n$ is semi-simple.

E 55)

Lemma:

If V is any G-representation, then

$V^{R_u(G)} = \{$ fixed vectors in V under $R_u(G)\}$

is a sub-G-representation.

Proof:

We have to show that $V^{R_u(G)}$ is fixed by any $g \in G$.

For this choose $v \in V^{R_u(G)}$ and $h \in R_u(G)$. By normality of $R_u(G) \subseteq G$ we can write $h \cdot g = g \cdot h'$ for some $h' \in R_u(G)$. Then:

$$h(g(v)) = (hg)(v) = (g h')(v) \overset{v \in V^{R_u(G)}!}{=} g(v) \Rightarrow g(v) \in V^{R_u(G)} \; \forall g.$$

Now assume V is a faithful simple representation of G.
We just proved that $V^{R_u(G)}$ is a subrepresentation. So we have two cases:

Case 1: $V^{R_u(G)} = (0)$

This cannot happen, because by Lie-Kolchin all representations of unipotent groups have non-trivial, fixed vectors.

Case 2: $V^{R_u(G)} = V$

Then any element in $R_u(G)$ acts trivially. As the representation is faithful, we get $R_u(G) = \{e\}$, i.e. G reductive.

Consider the canonical action of SO_n on K^n. This representation is faithful and for every $v \in K^n$, $v \neq 0$ there is some $\lambda \in K$ and $g \in SO_n$ with $v = \lambda \cdot g(e_1)$. Hence the representation is simple. $\Rightarrow$ SO_n reductive.

E 56)

Let us first deal with GL_n:

Fix $T = D_n \subseteq GL_n$ the diagonal torus. We choose the basis of $X(T) \cong \mathbb{Z}^n$ consisting of $e_1, \dots, e_n$ where e_i is the character given by the projection onto the (i,i)-th coeff.

Claim:

$$\mathcal{L}(G)_0 = \{\text{all diagonal matrices } \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \subseteq K^{n \times n}\}$$

and $\forall\, i \neq j$

$$\mathcal{L}(G)_{e_i - e_j} = \{\lambda \cdot E_{ij},\ \lambda \in K\}$$

for the matrix

$$E_{ij} = \begin{array}{c} \text{j-th column} \\ \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \end{array} \text{ i-th row}$$

Proof of the claim:

$\mathcal{L}(G)_0$ ~~$\subseteq$~~ consists of all matrices commuting with T, i.e. of all diagonal matrices.

Consider now $\mathcal{L}(G)_{e_i - e_j}$ for $i \neq j$: For $t = \mathrm{diag}(t_1, \dots, t_n) \in T$ we get:

$$Ad(t)(E_{ij}) = t \cdot E_{ij} \cdot t^{-1} = \begin{array}{cc} \text{j-th column} & \\ \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & t_i \cdot t_j^{-1} & \cdots \\ 0 & \vdots & 0 \end{pmatrix} & \text{i-th row} \end{array} =$$

$$= t_i \cdot t_j^{-1} \cdot E_{ij} = \underbrace{(e_i - e_j)}_{\text{viewed as a character!}}(t) \cdot E_{ij}$$

$$\Longrightarrow E_{ij} \in \mathcal{L}(G)_{e_i - e_j}.$$

Up to now, we only checked $\{\lambda E_{ij}\} \subseteq \mathcal{L}(G)_{e_i - e_j}$. But the E_{ij} (for $i \neq j$) and the diagonal matrices (in $\mathcal{L}(G)_0$) generate $\mathcal{L}(G)$ (as a vector space). Thus we have equality:

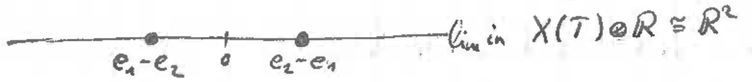
$$\mathcal{L}(G)_{e_i - e_j} = \{\lambda E_{ij}\}$$

and even $\mathcal{L}(G)_\chi = 0$ for $\chi \neq 0$ and $\chi \neq e_i - e_j$ $(i \neq j)$.

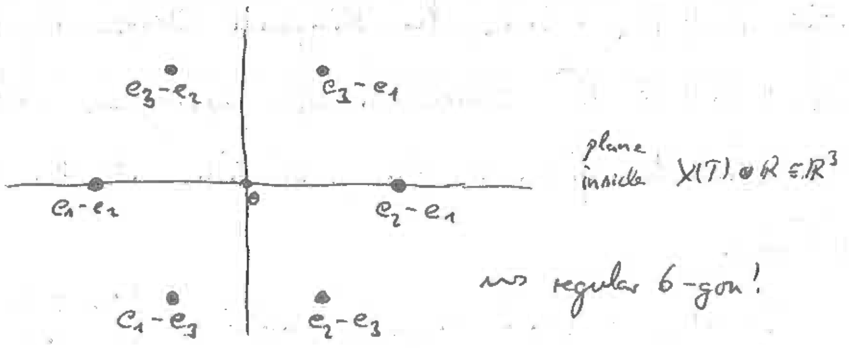
$\Longrightarrow$ Roots of GL_n are exactly the
$e_i - e_j$ for $i \neq j$.

Example: (after restriction to the $n-1$-dim. subspace given by $\sum \lambda_i = 0$) (but preserving the scalar product!)

$n = 2$:



$n = 3$:



⇝ regular 6-gon!

$n = 4$: Take a cube in $\mathbb{R}^3$ (center at the origin). Then the roots are precisely the midpoints of the edges of the cube!

Finally we treat the ~~case~~ group SL_n:

We fix the maximal torus $T' = T \cap SL_n = D_n \cap SL_n \subseteq SL_n$. Then $X(T') \subseteq X(T) \cong \mathbb{Z}^n$ has a basis given by $e_i - e_{i+1}$ (for $i = 1, \dots, n-1$).

Now for any $\chi \in X(T')$ we have:

$$\mathcal{L}(SL_n)_\chi = \mathcal{L}(GL_n)_\chi \cap \mathcal{L}(SL_n)$$

By our explicit calculation of $\mathcal{L}(GL_n)_\chi$ we see:

$$\mathcal{L}(SL_n)_\chi = \mathcal{L}(GL_n)_\chi \quad \text{if } \chi \neq 0$$

$$\mathcal{L}(SL_n)_0 = \mathcal{L}(GL_n)_0 \cap \mathcal{L}(SL_n) = \{A \in \mathrm{diag}(*, \dots, *) \subseteq K^{n \times n} \mid tr(A) = 0\}.$$

$\Rightarrow$ SL_n has the same roots as GL_n, but now viewed as elements in the $n-1$-dimensional lattice $X(T') \subseteq X(T)$!