Solutions to sheet 2

<u>E5</u>: We have $\mathrm{im}(f) = \{(x,y) \mid x \neq 0\} \cup \{(0,0)\}$ which follows from:

- If $x \neq 0$, then (x,y) has preimage $(x, y, \frac{y}{x})$.
- If $(x,y) = (0,0)$, then (x,y) has preimage $(0,0,0)$.
- If $(x,y) = (0,y)$ with $y \neq 0$, then any preimage $(0,y,q)$ must satisfy $0 = xz - y = -y$ ↯.

Now assume $\mathrm{im}(f)$ is locally closed, i.e. $\mathrm{im}(f) = U \cap Z$ for some open $U \subseteq K^2$ and some closed subset $Z \subseteq K^2$. Then

$$Z \supseteq \overline{\mathrm{im}(f)} = K^2$$

because $\mathrm{im}(f)$ contains the open subset $\{(x,y) \mid x \neq 0\} \subseteq K^2$, which is dense in K^2, because K^2 is irreducible.

$\Rightarrow Z = K^2$. and $\mathrm{im}(f) = U$.

It follows now that $K^2 \setminus \mathrm{im}(f) = \{(0,y), y \neq 0\}$ would be closed, or equivalently that $\{y \mid y \neq 0\} \subseteq K$ would be closed. Contradiction.

<u>E6</u>:

a) We refer to Algebra I for the proof that conjugation defines an action of an (abstract) group on itself. It remains to see, that

$$G \times X \to X, \quad (g, x) \mapsto g x g^{-1}$$

is a morphism of affine varieties. For this we need the following properties:

i) The inverse $i: G \to G$, $g \mapsto g^{-1}$ and the multiplication $m: G \times G \to G$ are morphisms of aff. varieties (by definition).

ii) The diagonal $\Delta: G \to G \times G$, $g \mapsto (g,g)$ is a morphism of affine varieties. Indeed a point $(x_1, \dots, x_n)$ is mapped to $((x_1, \dots, x_n), (x_1, \dots, x_n))$ which is a map given by (linear) polynomials, hence algebraic.

iii) Composition of morphisms of affine varieties are again morphisms.

ii) If $f: X \to Y$ is a morph. of aff. varieties, then so is
$f \times id: X \times Z \to Y \times Z$ (for any affine varieties X, Y, Z).
Then the conjugation action is given by:

$$G \times X \xrightarrow{\Delta \times id} G \times G \times X \cong G \times X \times G \xrightarrow{id \times id \times i} G \times X \times G \xrightarrow{m \times id} X \times G \xrightarrow{m} X$$
$$(g, x) \longmapsto (g, g, x) \mapsto (g, x, g) \longmapsto (g, x, g^{-1}) \longmapsto (gx, g^{-1}) \longmapsto gxg^{-1}.$$

b) Let $H \subseteq G$ be a finite normal subgroup. Then $H \subseteq X$ is stable under the conjugation action. Thus we may consider the alg. action
$G \times H \longrightarrow H$, $(g, h) \mapsto \{ghg^{-1}\}$.
Consider any $h \in H$ and the restriction of this morphism to
$G \cong G \times \{h\} \longrightarrow H$. , $g \mapsto ghg^{-1}$.
As G was assumed to be connected, the image of this morphism lies in one connected component of H, i.e. in one point of H (recall that finite subspaces of alg. varieties have the discrete topology!)
As we know the image of $e \in G$, this point has to be h again.
$\Rightarrow$ ~~$\forall g \in G$~~ $\forall h \in H: \forall g \in G: \ ghg^{-1} = h$
$\Rightarrow H$ lies in the center.

c) If H is not normal, consider
$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \subseteq GL_2$.
which lies not in the center.
If G needs not to be connected, let $H = G$ be any finite non-commutative group.

E7: We first show $f(H^{0})^0 = H^0$:
As f is an automorphism of varieties, it restricts to a homeo. of top. spaces $f: H^0 \to f(H^0) = f(H)^0$. In particular it follows
$\dim H^0 = \dim f(H)^0$.
Assume (for contradiction) that $f(H)^0 \subsetneq H^0$. Then by irreducibility of H^0, any sequence of irred. closed subsets $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n$ in $f(H)^0$

can be extended to a sequence $X_0 \subseteq X_1 \subseteq \dots \subseteq X_n \subsetneq H^0$ in H^0

But this would imply $\dim H^0 \geq \dim \rho(H)^0 + 1$ – Contradiction.

Thus we have $H^0 = \rho(H)^0$ and we may compute:

$$[H : H^0] \underset{\substack{\uparrow \\ \rho \text{ iso}}}{=} [\rho(H) : \rho(H^0)] = [\rho(H) : \rho(H)^0] = [\rho(H) : H^0]$$

But all values are finite (for algebraic groups!) and $\rho(H) \subseteq H$.

Thus $\rho(H) = H$ as desired.

E8:

a) Let GL_n act on K^n with the usual basis $e_1, \dots, e_n$. Then the set of vectors $v \in K^n$ that are eigenvectors for all elements in D_n is
$\Omega = \{\lambda \cdot e_i \mid \lambda \in K, i = 1, \dots, n\}$.

Then $\forall X \in N_{GL_n}(D_n)$, $A \in D_n$, $v \in \Omega$ we have:

$\exists \lambda \in K : X^{-1} A X v = \lambda v$

$\Rightarrow \exists \lambda \in K : A \cdot (Xv) = X(\lambda v) = \lambda \cdot (Xv)$

$\Rightarrow X \cdot v$ is again a common eigenvector of all elements in D_n

$\Rightarrow X$ fixes Ω, i.e. maps each e_i to some $\lambda_i e_{\sigma(i)}$,
where $\lambda_i \in K^\times$ and $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ is a permutation
(recall that X has to an element in GL_n!).

$\Rightarrow X$ is a product of a permutation matrix and a diagonal matrix, i.e. a monomial matrix.

b) First recall how to show that D_n is irreducible:

The canonical morphism $D_n \cong \mathbb{G}_m^n \xrightarrow{\sim} (K^\times)^n \subseteq K^n$ is a homeo. on its image. Thus D_n is homeomorphic to an open subset of the irreducible space K^n, hence irreducible itself.

Denote the set of permutation matrices by Perm ($\cong S_n$). Then multiplication with any element $P \in$ Perm defines an isomorphism of affine varieties

$P : N_{GL_n}(D_n) \longrightarrow N_{GL_n}(D_n)$.

Hence $P \cdot D_n \subseteq N_{GL_n}(D_n)$ is closed and irreducible (because it is homeomorphic to D_n!).

$\Rightarrow N_{GL_n}(D_n) = \bigcup_{P \in \mathfrak{S}_n} P \cdot D_n$ is a finite union of irred. subspaces.

As the $\{P \cdot D_n\}$ are not contained in each other, they form precisely the irreducible components, which (for alg. groups) coincide with connected components

$\Rightarrow N_{GL_n}(D_n)^0 = D_n$.