Solutions to sheet 3

E9:

a) Reflexivity: $G(x) < G(x)$ $\forall x$

True because $G(x) \subseteq \overline{G(x)}$.

Antisymmetry: If $G(x) < G(y)$ and $G(y) < G(x)$ then $G(x) = G(y)$

By assumption we have $G(x) \subseteq \overline{G(y)} \Rightarrow \overline{G(x)} \subseteq \overline{G(y)}$

and vice versa $\overline{G(y)} \subseteq \overline{G(x)} \Rightarrow \overline{G(x)} = \overline{G(y)}$.

By the lecture, $G(x)$ is open in $\overline{G(x)}$ (and automatically dense). The same is true for $G(y)$.

$\Rightarrow G(x) \cap G(y)$ is non-empty

$\Rightarrow G(x) = G(y)$.

Transitivity: $G(x) < G(y)$ and $G(y) < G(z) \Rightarrow G(x) < G(z)$.

True because $G(x) \subseteq \overline{G(y)} \subseteq \overline{G(z)}$.

b) An explicit calculation shows that the orbits are

$X_1 = K^\times \times K^\times \subseteq K^2$, $\quad X_2 = \{0\} \times K^\times \subseteq K^2$

$X_3 = K^\times \times \{0\} \subseteq K^2$, $\quad X_4 = \{0\} \times \{0\} \subseteq K^2$

(e.g. If $(x_1, y_1), (x_2, y_2) \in X_1$ are two points set $\lambda = \frac{x_2}{x_1}$ and $\mu \in K^2$ s.th. $\mu^2 = \frac{y_2}{y_1}$ (K alg. closed!). Then $(\lambda, \mu)((x_1, y_1)) = (x_2, y_2)$.).

We have $X_i < X_1$ $\forall i = 1, 2, 3, 4$,

$X_2 < X_2$, $\quad X_4 < X_2$

$X_3 < X_3$, $\quad X_4 < X_3$

and $X_4 < X_4$.

because $\overline{X_1} = K \times K$, $\overline{X_2} = \{0\} \times K$, $\overline{X_3} = K \times \{0\}$, $\overline{X_4} = X_4$.

This last claim can be seen in the following way:

Each $\overline{X_i}$ is closed and irreducible. Moreover $\overline{X_i} \setminus X_i$ is closed (by explicit calculation), hence X_i is open in $\overline{X_i}$ $\forall i$

$\Rightarrow X_i$ is dense in $\overline{X_i}$ $\Rightarrow \overline{X_i}$ is indeed the closure of X_i.

E 10

(a) $\forall x \in K^{3\times 3}$ and $\forall g \in GL_3$ we have

$$x^n = 0 \quad \text{if and only if} \quad (g(x))^n = (g\cdot x\cdot g^{-1})^n = g\cdot x^n\cdot g^{-1} = 0$$

$\Rightarrow Y$ is stable under the GL_3-action.

Moreover a matrix x in $K^{3\times 3}$ is nilpotent if and only if $x^3 = 0$. But this matrix equation translates into 9 polynomial equations in the coeff. of x. Hence the space of nilpotent matrices Y is given as the vanishing set of 9 polynomials in $K^{3\times 3}$.

(b) Jordan normal form classifies conjugacy classes. Hence we have 3 orbits in Y namely

$$Z_1 = GL_3 \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Z_2 = GL_3 \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Z_3 = GL_3 \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Claim: $Z_1 < Z_2 < Z_3$.

We only show $Z_1 < Z_2$, i.e. $Z_1 \subseteq \overline{Z_2}$. The other inclusion is done in a similar way. Consider the line

$$L = \left\{ \begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda \in K \right\} \subseteq K^{3\times 3}$$

Then Jordan normal form tells us that

$$L \cap Z_2 = \left\{ \begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda \in K^\times \right\}.$$

$\Rightarrow L \cap \overline{Z_2} = L$ (as the closure of $K^\times \subseteq K$ is all of K).

$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \overline{Z_2}$

$\Rightarrow Z_1 = GL_3 \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \subseteq \overline{Z_2}$.

E 11

Consider $G = \mathbb{Z}/2\mathbb{Z}$, $K = \mathbb{C}$, $X = \{0,1,2,3\} \subseteq K$.
with the action of G on X given by

$$0(x) = x \quad \text{for } 0 \in \mathbb{Z}/2\mathbb{Z},\ x \in X.$$

$$1(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \\ 3 & \text{if } x = 2 \\ 2 & \text{if } x = 3 \end{cases}$$

Assume now that the G-action extends to all of K.
Then the action of $1 \in \mathbb{Z}/2\mathbb{Z}$ is given by a polynomial $f \in K[x]$ s.th. $f(0) = 1$, $f(1) = 0$, $f(2) = 3$, $f(3) = 2$.
But any such polynomial has degree at least 3, hence cannot define an isomorphism $K \to K$! But for actions each elements acts via an isomorphism. Contradiction!

E 12

(a) Let $\Phi : \mathrm{Der}(S^{-1}R, S^{-1}R) \longrightarrow \mathrm{Der}(R, S^{-1}R)$, $\delta \mapsto \delta \circ i$.

Claim 1: Φ well-defined

$\delta \circ i$ is K-linear and additive as a composition of two such maps. Moreover $\forall x, y \in R$:

$$\delta \circ i(xy) = \delta(i(x) \cdot i(y)) = i(x) \cdot \delta(i(y)) + i(y) \cdot \delta(i(x))$$
$$= x \cdot \delta \circ i(y) + y \cdot \delta \circ i(x)$$

(as the action of x and $y \in R$ on $S^{-1}R$ are given via i).

Claim 2: Φ injective

Assume $\delta \in \mathrm{Der}(S^{-1}R, S^{-1}R)$ with $\Phi(\delta) = \delta \circ i \equiv 0$, i.e. $\delta(r) = 0$ $\forall r \in R$. Then $\forall r \in R$, $s \in S$:

$$0 = \delta(r) = \delta(s \cdot \tfrac{r}{s}) = s \cdot \delta(\tfrac{r}{s}) + \tfrac{r}{s} \cdot \delta(s) = s \cdot \delta(\tfrac{r}{s})$$

But $s \in (S^{-1}R)^{\times}$. Thus $\delta(\frac{r}{s}) = 0$.

$$\Rightarrow \delta(x) = 0 \quad \forall x \in S^{-1}R.$$

Claim 3: Φ surjective

For $\delta' \in \mathrm{Der}(R, S^{-1}R)$ define $\delta \in \mathrm{Der}(S^{-1}R, S^{-1}R)$ via

$$\delta(\tfrac{r}{s}) = \tfrac{1}{s}\delta'(r) - \tfrac{r}{s^2}\delta'(s) \quad \forall r \in R,\ s \in S.$$

Plugging in $S=1$ we see that $\delta' = \delta \circ i$ if δ is indeed a derivation. K-linearity is obvious. Additivity follows from

$$\delta\left(\frac{r}{s}+\frac{r'}{s'}\right) = \delta\left(\frac{rs'+sr'}{ss'}\right) =$$

$$= \frac{1}{ss'}\,\delta'(rs'+sr') - \frac{rs'+sr'}{(ss')^2}\,\delta'(ss') =$$

$$= \frac{1}{ss'}\left(r\delta'(s') + s'\delta'(r) + r'\delta'(s) + s\delta'(r')\right) -$$

$$- \frac{rs'+sr'}{s^2s'^2}\left(s'\delta'(s) + s\delta'(s')\right) =$$

$$= \frac{1}{s}\,\delta'(r) + \frac{1}{s'}\,\delta'(r') - \frac{r}{s^2}\,\delta'(s) - \frac{r'}{s'^2}\,\delta'(s') =$$

$$= \delta\left(\frac{r}{s}\right) + \delta\left(\frac{r'}{s'}\right).$$

Lastly we check that δ is well-defined, i.e. let $\frac{r}{s} = \frac{r'}{s'}$ in $S^{-1}R$.

$\Rightarrow \exists\, f \in R$ s.th. $f(rs'-sr') = 0$

$$\Rightarrow 0 = \delta'(f^2(rs'-sr')) = 2f(rs'-sr')\,\delta'(f) + f^2\,\delta(rs'-sr')$$

$$= f^2\left(s'\delta'(r) + r\delta(s') - s\delta'(r') - r'\delta'(s)\right)$$

Multiplication with $s\cdot s'$ gives:

$$0 = f^2\left(ss'^2\delta'(r) + rss'\delta'(s') - s^2s'\delta'(r') - r'ss'\delta'(s)\right) =$$

$$= f^2\left(ss'^2\delta'(r) + r's^2\delta'(s') - s^2s'\delta'(r') - rs'^2\delta'(s)\right) =$$

$$= f^2\left(s'^2\cdot\left(s\delta'(r) - r\delta'(s)\right) - s^2\left(s'\delta'(r') - r'\delta'(s')\right)\right)$$

$\Rightarrow$ In $S^{-1}R$ we have:

$$\delta\left(\frac{r}{s}\right) = \frac{s d'(r) - r d'(s)}{s^2} = \frac{s'\delta'(r') - r'\delta'(s')}{s'^2} = \delta\left(\frac{r'}{s'}\right)$$

as desired.

(b) Note: $\mathrm{Der}_K(K[G_m], K[G_m]) = \mathrm{Der}_K(K[x,x^{-1}], K[x,x^{-1}]) \overset{(a)}{\cong}$

$$\cong \mathrm{Der}_K(K[x], K[x,x^{-1}]).$$

- Each $\delta \in \mathrm{Der}_K(K[x], K[x,x^{-1}])$ is uniquely determined by $\delta(x) \in K[x,x^{-1}]$.
- For each $p \in K[x,x^{-1}]$, $\delta = p\cdot\frac{d}{dx}$ is a derivation with $\delta(x) = p \in K[x,x^{-1}]$

$$\Longrightarrow \mathrm{Der}_K(K[G_m], K[G_m]) = \left\{p\cdot\frac{d}{dx},\ p \in K[x,x^{-1}]\right\}.$$

(c) For any $\alpha \in K^*$ we have

$$\lambda_\alpha : K[\mathbb{G}_m] = K[x, x^{-1}] \longrightarrow K[\mathbb{G}_m] = K[x, x^{-1}]$$
$$x \longmapsto \alpha x.$$

$\Rightarrow$ For any $\delta = p \cdot \frac{d}{dx} \in \mathrm{Der}_K(\dots)$, $\alpha \in K^*$:

$$\delta \circ \lambda_\alpha(f) = p(x) \cdot \frac{d(f(\alpha x))}{dx} = p(x) \cdot \frac{d(\alpha x)}{dx} \cdot \frac{df}{dx}(\alpha x) =$$
$$= \alpha \cdot p(x) \cdot \frac{df}{dx}(\alpha x)$$

$$\lambda_\alpha \circ \delta(f) = \lambda_\alpha\left(p(x) \cdot \frac{df}{dx}(x)\right) = p(\cancel{\alpha}\,\alpha x) \cdot \frac{df}{dx}(\alpha x)$$

$\Rightarrow \alpha \cdot p(x) = p(\alpha x) \quad \forall \alpha \in K^*$

$\Rightarrow p(x) = \lambda x$ for some $\lambda \in K$.

$\Rightarrow$ ~~Lie~~ $\mathcal{L}(\mathbb{G}_m) = \mathrm{Der}(K[\mathbb{G}_m], K[\mathbb{G}_m])^{\mathbb{G}_m} = \{\lambda \cdot x \cdot \frac{d}{dx} \mid \lambda \in K\}$.