Solutions to sheet 4

E 13

(a) $[\delta_1, \delta_2]$ is additive and K-linear as a composition of such maps. Moreover $\forall x, y \in R$:

$$[\delta_1, \delta_2](xy) = \delta_1 \circ \delta_2(xy) - \delta_2 \circ \delta_1(xy) =$$
$$= \delta_1(x\delta_2(y) + y\delta_2(x)) - \delta_2(x\delta_1(y) + y\delta_1(x)) =$$
$$= x\delta_1\delta_2(y) + \delta_1(x)\delta_2(y) + y\delta_1\delta_2(x) + \delta_1(y)\delta_2(x) -$$
$$- x\delta_2\delta_1(y) - \delta_2(x)\delta_1(y) - y\delta_2\delta_1(x) - \delta_2(y)\delta_1(x) =$$
$$= x[\delta_1, \delta_2](y) + y[\delta_1, \delta_2](x).$$

(b) Let $R = K[x]$, $\delta_1 = \delta_2 = \frac{d}{dx} : K[x] \to K[x]$. (char $k \neq 2$)

$$\Rightarrow \delta_1 \circ \delta_2(x^2) = \delta_1(2x) = 2$$
$$2 \cdot x \cdot \delta_1 \circ \delta_2(x) = 2x\,\delta_1(1) = 0$$

$\lightning$ Product rule not satisfied.

Project 1

(a) As $\dim \mathrm{Der}_K(K[GL_n], K) = \dim T(GL_n, e) = \dim GL_n = n^2$ and $\dim \{\sum \lambda_{ij} \frac{\partial}{\partial x_{ij}} \mid \lambda_{ij} \in K\} = n^2$, it suffices to show that each $\frac{\partial}{\partial x_{ij}} \in \mathrm{Der}(K[GL_n], K)$.

This can be done by an explicit calculation, which I omit here (please ask, if you wish to see it!).

(b) $m^*(x_{ij}) = \sum_{k=1}^{n} y_{ik} z_{kj}$.

(c) Write $\delta_1 = \sum_{ij} \lambda_{ij} \frac{\partial}{\partial x_{ij}}$, $\delta_2 = \sum_{ij} \mu_{ij} \frac{\partial}{\partial x_{ij}}$

corresponding to the matrices

$$A = (\lambda_{ij})_{i,j}, \quad B = (\mu_{ij})_{i,j}$$

$$\rightsquigarrow [A, B] = \left(\sum_k \lambda_{ik}\mu_{kj} - \mu_{ik}\lambda_{kj}\right)_{i,j}$$

This matrix corresponds to the point derivation

$$\tilde{\delta} = \sum_{ij} \left(\sum_k \lambda_{ik}\mu_{kj} - \mu_{ik}\lambda_{kj}\right) \frac{\partial}{\partial x_{ij}}$$

We show that $\tilde{\delta} = [\delta_1, \delta_2]$.

- They agree on scalars: Indeed both sides are 0 then.

- They agree when evaluated at x_{ij}:

$$[\delta_1, \delta_2](x_{ij}) =$$
$$= (\delta_1 \otimes \delta_2 - \delta_2 \otimes \delta_1)\left(\sum_k y_{ik} z_{kj}\right) =$$
$$= \sum_k \left(\delta_1(y_{ik})\, \delta_2(z_{kj}) - \delta_2(y_{ik})\, \delta_1(z_{kj}) \right) =$$
$$= \sum_k \lambda_{ik} \mu_{kj} - \cancel{} \mu_{ik} \lambda_{kj} =$$
$$= \tilde{\delta}(x_{ij}).$$

- They agree on m_e^2 (for the max. ideal m_e).

$\tilde{\delta}$ is a point derivation, which implies:

$$\tilde{\delta}(m_e^2) = 0.$$

~~The~~ For $[\delta_1, \delta_2]$ the computation is slightly more complicated. Although a direct computation works as well, the following ~~th~~ trick simplifies the argument:

Denote: $p = \left(p(\underline{y}) \cdot q(\underline{z}) \text{ where } \cancel{p(\underline{z})}\ p, q \in m_e \right) \subseteq K[GL_n \times GL_n]$.

(i.e. the ideal generated by all

$y_{ij} \cdot z_{kl}$, $i \neq j,\ k \neq l$; $(y_{ii} - 1)\cdot(z_{kk} - 1)$, i, k bel.

$(y_{ii} - 1)\, z_{kl}$, i bel, $k \neq l$; $y_{ij}\,(z_{kk} - 1)$, $i \neq j$, k bel.).

Then a direct computation gives $\forall f \in m_e$:

$$m^*(f) = f(\underline{y}) + f(\underline{z}) \mod p.$$

Now note that $\delta_1 \otimes \delta_2$ (and $\delta_2 \otimes \delta_1$) vanish by construction on p^2.

Hence $\forall f, g \in m_e$:

$$(\delta_1 \otimes \delta_2)(m^*(fg)) = (\delta_1 \otimes \delta_2)(m^*(f) \cdot m^*(g)) =$$
$$= (\delta_1 \otimes \delta_2)\big((f(\underline{y}) + f(\underline{z}))(g(\underline{y}) + g(\underline{z}))\big) =$$
$$= \delta_1(f(\underline{y})g(\underline{y})) \cdot \delta_2(1) + \delta_1(f(\underline{y})) \cdot \delta_2(g(\underline{z})) +$$
$$+ \delta_1(g(\underline{y})) \cdot \delta_2(f(\underline{z})) + \delta_1(1) \cdot \delta_2(f(\underline{z})g(\underline{z})) =$$
$$= \delta_1(f) \cdot \delta_2(g) + \delta_1(g) \cdot \delta_2(f)$$

and similarly:

$$(\delta_2 \otimes \delta_1)(m^*(fg)) = \delta_1(f)\,\delta_2(g) + \delta_1(g)\,\delta_2(f)$$

$$\Rightarrow [\delta_1, \delta_2](f \cdot g) = 0.$$

But m_e^2 is generated by such elements. $\Rightarrow [\delta_1, \delta_2](m_e^2) = 0$.

The final thing to observe is, that $\tilde{\delta}$ and $[\delta_1, \delta_2]$ are K-linear morphisms and ~~the vector~~ $K[GL_n]$ is spanned (as a vector space) by 1, x_{ij} $(i,j \text{ bel})$ and the elements in $\mathfrak{m}_e^2$.

$\Rightarrow$ $\tilde{\delta} = [\delta_1, \delta_2]$ on all of $K[GL_n]$.

(in particular $[\delta_1, \delta_2]$ is indeed a point derivation).

E 14:

(a) We have: $i^*: K[GL_n] = K[x_{ij}][\frac{1}{\det}] \longrightarrow K[SL_n] = K[x_{ij}]/(\det)$

the canonical projection (mapping each x_{ij} again to x_{ij}).

Identifying $T(SL_n) = \mathrm{Der}(K[SL_n], K)$ and similarly for GL_n we get:

$$\begin{aligned} d_e i : \mathrm{Der}(K[SL_n], K) &\longrightarrow \mathrm{Der}(K[GL_n], K) \\ \delta &\longmapsto \delta \circ i^* \end{aligned}$$

Assume now $\delta \in \mathrm{Der}(K[SL_n], K)$ with $d_e i(\delta) = 0$.

$\Rightarrow$ $\delta \circ i^* = 0$

Surjectivity of $i^* \Rightarrow \delta = 0$

$\Longrightarrow$ $d_e i$ injective.

(b) Using the identification above, we get

$\mathrm{im}(d_e i) = \{ \delta \in \mathrm{Der}(K[GL_n], K) \mid \delta \text{ factors over } K[SL_n] \} =$

$= \{ \delta \in \mathrm{Der}(K[GL_n], K) \mid \delta(\det) = 0 \}$.

So, let's compute $\delta(\det)$:

If $\sigma \in S_n$, $\sigma \neq id$ is any non-trivial permutation, then

$$\delta\Big(\prod_{i=1}^{n} x_{i\sigma(i)}\Big) = \sum_{j=1}^{n} x_{1,\sigma(1)}(e) \cdot \ldots \cdot x_{j-1,\sigma(j-1)}(e) \cdot \delta(x_{j\sigma(j)}) \cdot \ldots \cdot x_{n\sigma(n)}(e) = 0$$

because there are at least two indices j s.th. $x_{j,\sigma(j)}(e) = 0$.

$\Rightarrow$ $\delta(\det) = \delta(x_{11} x_{22} \cdot \ldots \cdot x_{nn}) =$

$= \delta(x_{11}) + \delta(x_{22}) + \ldots + \delta(x_{nn})$

Thus using the description of $\mathcal{L}(GL_n)$ given in the project:

$\mathcal{L}(SL_n) = \mathrm{im}(d_e i) = \{ \delta \in \mathrm{Der}(K[GL_n], K) \mid \sum_{i=1}^{n} \delta(x_{ii}) = 0 \} =$

$= \{ A \in K^{n \times n} \mid tr(A) = 0 \}$.

E 15

(a) As in E14 it follows from the surjectivity of

$$i_1^* : K[GL_n \times GL_n] = K[y_{ij}, z_{ij}]\left[\tfrac{1}{\det(y)}, \tfrac{1}{\det(z)}\right] \longrightarrow K[GL_n] = K[y_{ij}]\left[\tfrac{1}{\det y}\right]$$

$$i_1^*(y_{ij}) = y_{ij}\,; \quad i_1^*(z_{ij}) = 0.$$

that $d_e i_1$ is injective. The same holds for $d_e i_2$.

Together with $\dim(T(GL_n) \oplus T(GL_n)) = 2n^2 = \dim(T(GL_n \times GL_n))$

this implies, that it suffices to show:

Claim: $\operatorname{im}(d_e i_1) \cap \operatorname{im}(d_e i_2) = 0$.

Proof of claim:

Assume $\delta \in \operatorname{Der}(K[GL_n \times GL_n], K)$ lies in this intersection.

As $\delta \in \operatorname{im}(d_e i_1)$, δ has to factor through $K[y_{ij}][\frac{1}{\det y}]$.

In other words $\delta(z_{ij}) = 0 \quad \forall i, j$.

On the other hand $\delta \in \operatorname{im}(d_e i_2)$ implies by a similar

argument $\delta(y_{ij}) = 0 \quad \forall i, j$.

$\Longrightarrow \delta \equiv 0$ is the zero derivation.

$\Longrightarrow \operatorname{im}(d_e i_1) \cap \operatorname{im}(d_e i_2) = 0$.

(b) We have $m \circ i_1 = id_{GL_n}$. Thus taking derivations:

$$id_{T(GL_n)} = d_e\, id_{GL_n} = d_e(m \circ i_1) = d_{e,e}(m) \circ d_e i_1 : T(GL_n) \to T(GL_n).$$

Reformulated:

$$\begin{tikzcd} T(GL_n) \arrow[d, "d_e i_1"'] \arrow[dr, "d_e id_{GL_n}"] & \\ T(GL_n \times GL_n) \arrow[r, "d_{e,e} m"] & T(GL_n) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, mapsto] \arrow[dr, mapsto] & \\ (A, 0) \arrow[r, dashed, mapsto, "d_{e,e} m"] & A \end{tikzcd}$$

Similarly for $m \circ i_2$ gives:

$$d_{e,e} m(0, B) = B$$

But $d_{e,e} m$ is a linear morphism, hence already defined by

$d_{ee} m(A, 0) = A$ and $d_{ee} m(0, B) = B$.

$\Longrightarrow d_{e,e} m(A, B) = A + B \quad \forall A, B.$

(c) Consider the morphism given in the hint:

$$GL_n \xrightarrow[\text{diag.}]{\Delta} GL_n \times GL_n \xrightarrow{(id, \iota)} GL_n \times GL_n \xrightarrow{m} GL_n.$$

As it has constant image the unit matrix, we get:

$$0 = d_e(m \circ (id, \iota) \circ \Delta) =$$

$$= d_{e,e}(m) \circ d_{e,e}(id, \iota) \circ d_e \Delta$$

From its description on coordinate rings one gets

$$d_e \Delta : T(GL_n) \longrightarrow T(GL_n) \oplus T(GL_n) = T(GL_n \times GL_n)$$
$$A \longmapsto (A, A).$$

$$\Rightarrow 0 = d_{e,e}(m) \circ d_{ee}(id, \iota) \circ d_e \Delta (A) =$$

$$= d_{e,e}(m) \circ d_{e,e}(id, \iota)(A, A) =$$

$$= d_{e,e}(m)(A, d_{e}\iota(A)) \overset{(c)}{=}$$

$$= A + d_e \iota(A) \qquad \forall A \in T(GL_n)$$

$$\Rightarrow d_e \iota(A) = -A.$$

(d) The transpose ^{T} is given on coordinate rings by

$$T^* : K[GL_n] = K[x_{ij}][\tfrac{1}{\det}] \longrightarrow K[x_{ij}][\tfrac{1}{\det}] = K[GL_n]$$

$$x_{ij} \longmapsto x_{ji}.$$

Hence for any $\delta = \sum \lambda_{ij} \frac{\partial}{\partial x_{ij}} \in T(GL_n)$ we have

$$d_e^T(\delta) = \delta \circ T^* = \sum \lambda_{ij} \frac{\partial}{\partial x_{ji}} = \sum \lambda_{ji} \frac{\partial}{\partial x_{ij}}.$$

In terms of matrices this translates as:

$$d_e^T(A) = A^T.$$

E16

Consider the morphism

$$f: GL_n \xrightarrow{\Delta} GL_n \times GL_n \xrightarrow{(id, {}^T)} GL_n \times GL_n \xrightarrow{m} GL_n$$
$$A \longmapsto (A, A) \longmapsto (A, A^T) \longmapsto A \cdot A^T$$

Its derivation is by E15:

$$d_e f(A) = d_{e,e} m \circ d_{e,e}(id, {}^T)(A, A) =$$

$$= d_{e,e} m (A, A^T) = A + A^T$$

Now consider: $O_n \xhookrightarrow{i} GL_n \xrightarrow{f} GL_n$.

It has constant image, hence vanishing derivation.

$$0 = d_e f \circ d_e i\,(A) = d_e i(A) + d_e i(A)^T.$$

$$\Longrightarrow \quad \mathcal{L}(O_n) = T(O_n) \subseteq \{A \in T(GL_n) \mid A + A^T = 0\} \subseteq \mathcal{L}(GL_n).$$

Now we compare dimension: By the remark:

$$\dim \mathcal{L}(O_n) = \dim O_n = \frac{1}{2} n(n-1)$$

and on the other hand:

$$\dim \{A \in K^{n\times n} \mid A + A^T = 0\} \overset{\operatorname{char} K \neq 2}{=} \frac{1}{2} n(n-1)$$

Hence the inclusion above was already an equality!.