Solutions to sheet 5

E 17

(a) Let D_n be the subgroup of all diagonal matrices and consider

$$f : T_n \longrightarrow D_n \quad , \quad \begin{pmatrix} a_{11} & \cdots & a_{ij} \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} a_{11} & \cdots & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}.$$

Then using $a_{ii} \neq 0 \ \forall i$ we get:

$$f^{-1}\begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} = \left\{ \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \right\} = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \cdot U_n .$$

$\Rightarrow T_n/U_n \cong D_n$ is an affine variety isomorphic to $(K^\times)^n$.

(b) Identify $\mathbb{P}^1$ with the set of lines in K^2 and fix the line $L_0 = K \cdot \binom{1}{0} \subseteq K^2$. Consider now the morphism

$$f : SL_2 \longrightarrow \mathbb{P}^1 \quad , \quad A \longmapsto A \cdot L_0 .$$

Then f is surjective and

$$f^{-1}(L_0) = \{ A \in SL_2 \mid A \cdot L_0 = L_0 \} = T$$

Hence we obtain a bijection

$$f : SL_2/T \xrightarrow{\ \sim\ } \mathbb{P}^1 .$$

E 18

~~(i) $\Rightarrow$ (ii): Consider G/H with the G-action by left multiplication. Let $\{Z_i\} \subseteq G$ be the irred. components with $Z_0 = G^0$ the identity component.~~

(i) $\Rightarrow$ (iii)

Assume that $H \cdot G^0 \subsetneq G$ is a proper subgroup. As it is contains G^0 it is of finite index in G. Let now $X = \coprod_{\alpha \in G/H\cdot G^0} p_\alpha$ be a variety consisting of a finite disjoint union of ~~pts~~ (closed) points indexed by the cosets $\alpha \in G/HG^0$. Consider the map

$$\varphi : G \longrightarrow X \quad , \quad g \longmapsto p_\alpha \text{ if } g \text{ lies in the coset } \alpha \in G/HG^0 .$$

This is a morphism of varieties, because each connected component of G gets contracted to one single point and such a contraction is a morphism of varieties. Moreover we have by definition

$$\varphi(g \cdot h) = \varphi(g) \quad \forall g \in G, h \in H.$$

Hence by the universal property of quotients we get a commutative diagram:

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "\varphi"'] & G/H \arrow[d, dashed, "\exists\,\tilde{\varphi}"] \\ & X \end{tikzcd}$$

As φ is surjective, so is $\tilde{\varphi}$. But there cannot be a surjection between an irreducible space (like G/H) and one with many irreducible components (like X). Contradiction.

(iii) $\Rightarrow$ (ii)

Let $Z \subseteq G$ be any connected component and pick some element $z \in Z$. By (iii) we may write $z = g \cdot h$ for $g \in G^\circ$, $h \in H$ (use $G^\circ \subseteq G$ normal for this).

$\Rightarrow$ $h = g^{-1} z \in Z \cap H$.

(ii) $\Rightarrow$ (i)

We claim that $G^\circ \hookrightarrow G \to G/H$ is surjective. Pick for this any $[z] \in G/H$ and any preimage $z \in G$. Then z lies in some connected component Z. By (ii) we can find $h \in Z \cap H$. As G° acts transitively on Z, we may therefore write

$z = g \cdot h$ for some $g \in G^\circ$

$\Rightarrow$ $[z] = [g] \in G/H$ and $[g]$ lies in the image of G°.

Thus G/H is the image of the irreducible space G°, hence irreducible itself.

<u>E 19</u>

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}^{-1} \quad \text{(cf. lin. alg.)}$$

Decomposing the matrix in the middle, we get the additive Jordan decomposition as

$$g_s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & -1 \end{pmatrix}$$

$$g_n = \begin{pmatrix} 1&0&0\\0&0&1\\0&1&-1 \end{pmatrix} \cdot \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0\\0&0&1\\0&1&-1 \end{pmatrix}^{-1} = \begin{pmatrix} 0&1&1\\0&0&0\\0&0&0 \end{pmatrix}.$$

From this we get the multiplicative Jordan decomposition by setting

$$g_u = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} + g_s^{-1} \cdot g_n = \begin{pmatrix} 1&-1&-1\\0&1&0\\0&0&1 \end{pmatrix}$$

Although we did all computations in the space of all matrices, g_s and g_u lie (as predicted by the general theory) in SL_3.

E20

(a) $\begin{pmatrix} 1&1\\0&1 \end{pmatrix}$ and $\begin{pmatrix} 1&0\\1&1 \end{pmatrix} \in SL_{2u}$

but $\begin{pmatrix} 2&1\\1&1 \end{pmatrix} = \begin{pmatrix} 1&1\\0&1 \end{pmatrix} \cdot \begin{pmatrix} 1&0\\1&1 \end{pmatrix} \notin SL_{2,u}$

Let $\lambda \in K$, $\lambda \neq 0, \pm 1$ be any element. Then

$\begin{pmatrix} \lambda&1\\0&1/\lambda \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{\lambda}&0\\0&\lambda \end{pmatrix} \in SL_{2,s}$

but $\begin{pmatrix} \lambda&1\\0&1/\lambda \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\lambda}&0\\0&\lambda \end{pmatrix} = \begin{pmatrix} 1&\lambda\\0&1 \end{pmatrix} \notin SL_{2,s}$.

(b) From the lecture we know that SL_{2u} is closed. Now $SL_{2u} \neq SL_2$ and the only open and closed subspace of the irreducible variety SL_2 is SL_2 itself. Hence $SL_{2,u}$ cannot be open.

(c) Consider $f: X = K \setminus \{0\} \longrightarrow SL_2$

$$\lambda \longmapsto \begin{pmatrix} \lambda&1\\0&\lambda^{-1} \end{pmatrix}$$

and assume SL_{2s} is closed. Then by continuity of f, the set

$$K \setminus \{0, \pm 1\} = f^{-1}(SL_{2,s}) \subseteq X = K \setminus \{0, \pm 1\}$$

would be closed as well. But it is surely open and X irreducible, giving the desired contradiction.

(d) Let $Z = SL_2 \setminus SL_{2s}$. We show that Z is not closed. Consider

$g: K \longrightarrow SL_2$, $\lambda \longmapsto \begin{pmatrix} 1&\lambda\\0&1 \end{pmatrix}$.

Then $g^{-1}(Z) = K \setminus \{0\}$ ~~as~~ which is not closed ~~open~~. Hence Z cannot be closed as well.