Solutions to sheet 6

E21: Let $K = \mathbb{C}$, $\alpha \in \mathbb{C}$, $\alpha \neq 0$ any element. Let
$H = \left\{ \begin{pmatrix} \alpha^n & n\alpha^n \\ 0 & \alpha^n \end{pmatrix}, n \in \mathbb{Z} \right\} \subseteq GL_2$.
Then we have the Jordan decomposition
$$\begin{pmatrix} \alpha & \alpha \\ & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
But neither $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ nor $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ lies in H.

E22:

(a) After an embedding $G \subseteq GL_n$ we may view g as a matrix of finite order in GL_n. Then $g^n - \mathbb{1} = 0$ for some $n \gg 0$. In particular the minimal polynomial of g ~~is~~ divides $x^n - 1$, hence has no multiple zeroes.

$\Rightarrow$ g is diagonalizable in GL_n

$\Rightarrow$ $g \in G$ is semi-simple.

(b) Consider the identity component $G^\circ \subseteq G$. It is a normal subgroup of finite index in G. Then consider
$$\pi : G \longrightarrow G/G^\circ$$
By (a) we know that the elements in G/G° are all semi-simple and by a result from the lecture we know that they are unipotent as well (as images of unipotent elements under a group morphism). But the unipotent and semi-simple elements intersect only in the identity:
$$G/G^\circ = \{e\}$$
$\Rightarrow$ $G = G^\circ$ is connected.

E23: ||| Warning: p, q depend on g!! Thus the arguments below are completely nonsense!! |||

We first assume $G = GL_n$. By lemma 5.2 b) from the lecture, there are polynomials $p, q \in K[x]$ s.th.
$$p(g) = g_s \quad \text{and} \quad q(g) = g_u \quad \forall g \in GL_n.$$

In particular

$$\varphi_s : GL_n \longrightarrow GL_n \quad , \quad g \longmapsto g_s = p(g)$$

$$\varphi_n : GL_n \longrightarrow K^{n\times n} \quad , \quad g \longmapsto g_n = \varphi(g)$$

are morphisms of affine varieties.

Now we can write $\varphi_u : GL_n \longrightarrow GL_n$ as the composition

$$\begin{aligned}
\varphi_u : GL_n &\xrightarrow{\text{diag}} GL_n \times GL_n \xrightarrow{(\varphi_s, \varphi_n)} GL_n \times K^{n\times n} \xrightarrow{(\mathrm{inv}, \mathrm{id})} GL_n \times K^{n\times n} \xrightarrow{m} \\
g &\longmapsto (g, g) \longmapsto (g_s, g_n) \longmapsto (g_s^{-1}, g_n) \longmapsto
\end{aligned}$$

$$\begin{aligned}
&\xrightarrow{m} K^{n\times n} \xrightarrow{+\mathbb{1}_n} K^{n\times n} \\
&\longmapsto g_s^{-1} \cdot g_n \longmapsto \mathbb{1}_n + g_s^{-1} g_n = g_u .
\end{aligned}$$

$\Longrightarrow \varphi_u : GL_n \longrightarrow GL_n$ is ~~g~~ a morphism of aff. varieties.

$\varphi_s : G \longrightarrow G$, $\varphi_u : G \longrightarrow G$ are just the restrictions of the morphisms above to G, hence they are morphisms of aff. var. as well.

$\Longrightarrow (\varphi_s, \varphi_u) : G \longrightarrow G_s \times G_u$ is a morphism of affine varieties.

This is by construction the inverse of the multiplication morphism (in the sense of affine varieties)

$$m : G_s \times G_u \longrightarrow G .$$

But m is a group morphism (cf. e.g. lecture). Hence $(\varphi_s, \varphi_u) = m^{-1}$ has to be a group morphism as well.

E 24

Let $A = \left\{ id , \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, $B = \left\{ id , \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$. Then:

$$\begin{aligned}
(A, B) &= \left\langle \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \right\rangle = \\
&= \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle = \\
&= \left\{ \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} , n \in \mathbb{Z} \right\} .
\end{aligned}$$

Thus $(A, B) \subseteq GL_2$ is an infinite discrete subgroup. Hence it cannot be algebraic.

Comments on E23:

Claim 1:

The map $\varphi_s : SL_2 \longrightarrow SL_2$, $g \longmapsto g_s$

is not a morphism of ~~algebraic groups~~ affine varieties

Proof:

Consider the set of semi-simple elements $SL_{2,s} \subseteq SL_2$. Any element in $SL_2 \setminus SL_{2,s}$ has only one ~~eg~~ eigenvalue occuring twice. Hence this eigenvalue is ~~either~~ ± 1. In particular under the map

$$tr : SL_2 \longrightarrow K, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a+d$$

all elements in $SL_2 \setminus SL_{2,s}$ map to either 2 or -2.

$\Longrightarrow$ $tr^{-1}(K \setminus \{\pm 2\}) \subseteq SL_2$ is open and contained in $SL_{2,s}$

$\Longrightarrow$ φ_s is the identity on the open dense subset

$$tr^{-1}(K \setminus \{\pm 2\}) \subseteq SL_2$$

If φ_s would be a morphism of affine varieties, then

$$\varphi_s = id \text{ on all of } SL_2.$$

But this is clearly nonsense.

Claim 2:

If G is commutative, then φ_s and φ_u are morphisms of affine varieties.

Proof:

By proposition 5.5. there is an embedding

$\rho : G \hookrightarrow GL_n$

s.th. • the image lies in the set of upper triangular matrices T_n

• every semi-simple element is mapped to D_n.

Consider now $f : T_n \longrightarrow D_n$, $\begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \longmapsto \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}$.

We claim that $\varphi_s = f|_G$.

To show this consider any $g \in G$ with semi-simple part g_s.

Then $\rho(g_s)$ is again semi-simple. Hence we can write

$$\rho(g) = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix}, \quad \rho(g_s) = \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{pmatrix}.$$

for some $a_i, b_i \in K \setminus \{0\}$.

Then $g \cdot g_s^{-1} = g_u$ is unipotent, hence so is

$$\rho(g) \cdot \rho(g_s)^{-1} = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \cdot \begin{pmatrix} b_1^{-1} & & 0 \\ & \ddots & \\ 0 & & b_n^{-1} \end{pmatrix} = \begin{pmatrix} a_1 b_1^{-1} & & * \\ & \ddots & \\ & & a_n b_n^{-1} \end{pmatrix}$$

But the unipotent elements in T_n are U_n and hence

$a_i = b_i \ \forall i$

and indeed $\varphi_s(g) = f(g)$.

In particular this gives that φ_s is a morphism of affine varieties.

Finally we can conclude that

$$\begin{aligned} \varphi_u : G &\xrightarrow{\text{diag}} G \times G \xrightarrow{(id, \varphi_s)} G \times G \xrightarrow{(id, (\cdot)^{-1})} G \times G \xrightarrow{m} G \\ g &\longmapsto (g, g) \longmapsto (g, g_s) \longmapsto (g, g_s^{-1}) \longmapsto g \cdot g_s^{-1} = g_u \end{aligned}$$

is a morphism of affine varieties as well.