Solutions to sheet 7.

E25

(a) Let $g \in G$. Then it suffices to check $g(N,N')g^{-1} \subseteq (N,N')$ on a set of generators of (N,N'). We use for this the set $\{n \cdot n' \cdot n^{-1} \cdot n'^{-1},\ n \in N, n' \in N'\}$.

But there one computes:

$$g(nn'n^{-1}n'^{-1})g^{-1} = (gng^{-1})(gn'g^{-1})(gn^{-1}g^{-1})(gn'^{-1}g^{-1}) = \qquad (*)$$
$$= [gng^{-1}, gn'g^{-1}] \in (N,N').$$

(b) To show $(H, N_G(H)) \subseteq H$ it suffices to check this on generators. Hence pick $h \in H$, $n \in N_G(H)$. Then:

$$[h,n] = \underbrace{h}_{\in H} \cdot \underbrace{(n \cdot h^{-1} \cdot n^{-1})}_{\in H} \in H$$
$$\Longrightarrow (H, N_G(H)) \subseteq H.$$

Let now $N \leq G$ s.th. $(H,N) \subseteq H$. Then $\forall n \in N, h \in H$:

$$n \cdot h \cdot n^{-1} = \underset{H}{\overset{\in}{h}} \cdot \underbrace{[h^{-1}, n]}_{\in H} \in H$$

see comments on last page for this implication

$$\Longrightarrow n \in N_G(H)$$
$$\Longrightarrow N \leq N_G(H)$$
$$\Longrightarrow N_G(H) \text{ is the largest subgroup s.th. } (H,N) \subseteq H.$$

(c) Let $n \in N_G(H)$, $h, h' \in H$. Then:

$$n[h,h']n^{-1} \overset{(*)}{=} [\underbrace{nhn^{-1}}_{\in H}, \underbrace{nh'n^{-1}}_{\in H}] \in (H,H).$$
$$\Rightarrow n \in N_G((H,H)).$$

For the second part take $G = GL_2$, $H = D_2$ the diagonal matrices.

$\Rightarrow (H,H) = \{e\}$ (as D_2 is commutative)

$\Rightarrow N_{GL_2}((H,H)) = GL_2$

but $N_{GL_2}(H) = N_{GL_2}(D_2) \neq GL_2$ as $D_2 \leq GL_2$ not normal.

(d) Induction on i, starting with $i=0$, when the statement is trivial.

Thus assume $f(\mathcal{D}^i(G)) \subseteq \mathcal{D}^i(G')$. Then $\forall g, g' \in \mathcal{D}^i(G)$:

$$f([g,g']) = [f(g), f(g')] \in \mathcal{D}(\mathcal{D}^i(G')) = \mathcal{D}^{i+1}(G').$$

But the $[g,g']$ generate $\mathcal{D}^{i+1}(G)$. Hence

$$f(\mathcal{D}^{i+1}(G)) \subseteq \mathcal{D}^{i+1}(G').$$

Assume now $f(\mathcal{C}^i(G)) \subseteq \mathcal{C}^i(G')$. Then let $g \in \mathcal{C}^i(G)$, $g' \in G$.

$$f([g,g']) = [\underset{\mathcal{C}^i(G')}{\underset{\cap}{f(g)}}, \underset{G'}{\underset{\cap}{f(g')}}] \in \mathcal{C}(\mathcal{C}^i(G')) = \mathcal{C}^{i+1}(G').$$

$$\Rightarrow f(\mathcal{C}^{i+1}(G)) \subseteq \mathcal{C}^{i+1}(G').$$

(e) Proceed again by induction on i, starting with the trivial $i=0$.

Assume $f(\mathcal{D}^i(G)) = \mathcal{D}^i(G')$. Let $g, g' \in \mathcal{D}^i(G')$ and pick preimages $\tilde{g}, \tilde{g}' \in \mathcal{D}^i(G)$ under f. Then

$$f([\tilde{g}, \tilde{g}']) = [f(\tilde{g}), f(\tilde{g}')] = [g, g'].$$

$\Rightarrow$ All the generators $[g,g']$ of $\mathcal{D}^{i+1}(G')$ lie in the image of $\mathcal{D}^{i+1}(G)$.

$$\Rightarrow f(\mathcal{D}^{i+1}(G)) = \mathcal{D}^{i+1}(G').$$

Same arguments for $\mathcal{C}^i(G)$.

<u>E26</u>:

Let $T_2 = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq GL_2$ with normal subgroup $U_2 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \subseteq T_2$.

$\Rightarrow U_2$ and $T_2/U_2 \cong D_2$ are nilpotent (because commutative)

But T_2 is not nilpotent (cf. lecture).

<u>E27</u>:

(a) $Z(G/Z_i(G))$ is normal in $G/Z_i(G)$.

Moreover preimages of normal subgroups under group morphisms are again normal

$\Rightarrow Z_{i+1}(G)$ is normal in G.

(b) As G is nilpotent, we may choose $n \in \mathbb{N}$ s.th. $\mathcal{C}^n(G) = \{e\}$.

We claim now: $Z_i(G) \supseteq \mathcal{C}^{n-i}(G) \quad \forall i = 0, \dots, n$.

By definition this is true for $i = 0$. Assume by induction that $\mathcal{C}^{n-i}(G) \subseteq Z_i(G)$. Now:

$$[g, c] \in \mathcal{C}^{n-i}(G) \quad \forall g \in G, \ c \in \mathcal{C}^{n-i-1}(G).$$

$$\Rightarrow g \cdot c = c \cdot g \in G/\mathcal{C}^{n-i}(G) \quad \forall g \in G, \ c \in \mathcal{C}^{n-i-1}(G).$$

Induction

$$\Rightarrow g \cdot c = c \cdot g \in G/Z_i(G) \quad \forall g \in G, \ c \in \mathcal{C}^{n-i-1}(G).$$

$\Rightarrow$ c lies in the center of $G/Z_i(G) \quad \forall c \in \mathcal{C}^{n-i-1}(G)$

$$\Rightarrow \mathcal{C}^{n-i-1}(G) \subseteq \pi_i^{-1}(Z(G/Z_i(G))) = Z_{i+1}(G).$$

But for $i = n$ the claim states nothing else than

$$G = \mathcal{C}^0(G) \subseteq Z_n(G)$$

and hence equality (as $Z_n(G) \subseteq G$ is trivial).

(c) This time we claim again $\mathcal{C}^i(G) \subseteq Z_{n-i}(G) \quad \forall i = 0, \dots, n$ and prove it again by induction on i (but note the difference in the direction of the indices we deal with).

$i = 0$ is again trivial, so assume $\mathcal{C}^i(G) \subseteq Z_{n-i}(G)$.

$$\Rightarrow \pi_{n-i-1}(\mathcal{C}^i(G)) \subseteq Z(G/Z_{n-i-1}(G)).$$

$$\Rightarrow g \cdot c = c \cdot g \in G/Z_{n-i-1}(G) \quad \forall c \in \mathcal{C}^i(G), \ g \in G.$$

$$\Rightarrow [g, c] = 1 \in G/Z_{n-i-1}(G) \quad \forall c \in \mathcal{C}^i(G), \ g \in G.$$

$$\Rightarrow \pi_{n-i-1}(\mathcal{C}^{i+1}(G)) = \{1\} \subseteq G/Z_{n-i-1}(G)$$

$$\Rightarrow \mathcal{C}^{i+1}(G) \subseteq Z_{n-i-1}(G).$$

But for $i = n$ we again have the desired result:

$$\mathcal{C}^n(G) \subseteq Z_0(G) = \{e\}.$$

$\Rightarrow$ G nilpotent.

E 28:

Claim 1: f^n is trivial for some $n \gg 0$

Proof of claim 1:

As $f: G \longrightarrow G$, $x \mapsto [g, x]$ we have restrictions

$f|_{C^i(G)}: C^i(G) \longrightarrow C^{i+1}(G)$.

Hence $f^n: G \longrightarrow C^n(G)$. Thus for sufficiently large $n \in \mathbb{N}$, f^n maps everything to the identity element.

Claim 2: $d_e f = Ad(g) - id_{\mathcal{L}(G)}$.

Proof of claim 2:

Decompose:

$$f: G \xrightarrow{\text{diag}} G \times G \xrightarrow{(\text{Inn}(g), ()^{-1})} G \times G \xrightarrow{m} G$$

$$x \longmapsto (x, x) \longmapsto (gxg^{-1}, x^{-1}) \longmapsto gxg^{-1}x^{-1}.$$

$$\Rightarrow d_e f(\delta) = d_{e,e} m \circ (d_e \text{Inn}(g), d_e()^{-1})(\delta, \delta) =$$

$$= d_{e,e} m(Ad(g)(\delta), -\delta) =$$

$$= Ad(g)(\delta) - \delta.$$

Using both claims and the product rule we get for $n \gg 0$:

$$0 = d_e(f^n) = (d_e f)^n = (Ad(g) - id_{\mathcal{L}(G)})^n$$

$\Rightarrow Ad(g) - id_{\mathcal{L}(G)} \in GL(\mathcal{L}(G))$ is nilpotent

$\Rightarrow Ad(g) \in GL(\mathcal{L}(G))$ is unipotent.

On the other hand $Ad(g)$ is the image of the semi-simple element $g \in G$ under $Ad: G \longrightarrow GL(\mathcal{L}(G))$.

$\Rightarrow Ad(g)$ is semi-simple

But the only element that is both unipotent and semi-simple is the identity element

$\Rightarrow Ad(g) = id_{\mathcal{L}(G)}$.

Comments on E 25 b)

Claim 1:

There exists an (abstract) group G, a subgroup $H \leq G$ and an element $n \in G$ with $nHn^{-1} \subsetneq H$ (but not equality).

Proof:

Let G_0 be any group. Then the group $\mathbb{Z}$ acts on $G_0^{\mathbb{Z}} = \prod_{i \in \mathbb{Z}} G_0$ by shifting elements (i.e. $j \in \mathbb{Z}$ acts on $(g_i) \in G_0^{\mathbb{Z}}$ by $j \cdot (g_i) = (g_{i+j})$). Thus we can form the semi-direct product $G = G_0^{\mathbb{Z}} \rtimes \mathbb{Z}$. Consider $n = ((e)_{i \in \mathbb{Z}}, 1)$ and

$$H = \left(\prod_{i \geq 0} G_0\right) \rtimes \{0\} \subseteq G_0^{\mathbb{Z}} \rtimes \mathbb{Z}.$$

Then one computes explicitly:

$$nHn^{-1} = \left(\prod_{i \geq 1} G_0\right) \rtimes \{0\} \subsetneq H.$$

Claim 2:

If G is any group, $H \leq G$ a subgroup, $N \leq G$ another subgroup s.th.

$$nHn^{-1} \subseteq H \quad \forall n \in N \qquad (*)$$

Then even $nHn^{-1} = H \;\; \forall n \in N$.

Proof:

Assume $nHn^{-1} \neq H$ for some $n \in N$. Then pick some $h \in H \setminus nHn^{-1}$. Applying $(*)$ to n^{-1} we get

$$n^{-1}hn \in H.$$

Applying $(*)$ again to n we now get

$$n(n^{-1}hn)n^{-1} = h \in H$$

Contradiction.