Solutions to sheet 8

E29)

The finite group $G = S_3$ is solvable and acts on K^3 by permuting the basis vectors e_1, e_2, e_3. As $e_1+e_2+e_3$ is a common eigenvector, consider the induced representation ρ on $V = K^3/\langle e_1, e_2, e_3\rangle$.

(i.e. we consider the representation $(1\,2) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $(2\,3) \mapsto \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$).

Then $\rho((12))$ has only eigenvectors $K^* \cdot \overline{(e_1+e_2)}$ and $K^* \cdot \overline{(e_1-e_2)}$ in V. But none of them is an eigenvector of $\rho((23))$.

Thus this is the desired counterexample.

E30)

Assume first that G is connected and solvable. Then we may embed $G \hookrightarrow T_n$ (for some n). Then

$$(G, G) \hookrightarrow (T_n, T_n) = U_n$$

and U_n is unipotent. Hence (G, G) is unipotent as a subgroup of a unipotent group.

Assume that (G, G) is unipotent. Then (G, G) is in particular nilpotent and hence solvable.

$\Rightarrow \exists n > 0: \mathcal{D}^n((G, G)) = \{e\}$.

As $\mathcal{D}^i((G, G)) = \mathcal{D}^{i+1}(G)$ $\forall i$ we get $\mathcal{D}^{n+1}(G) = \{e\}$ and G is solvable.

E31)

For $G = T_n$ this was already discussed in some exercise class (but feel free to contact me, if you wish to see details!).

For general G, embed $G \hookrightarrow T_n$ and choose a sequence for T_n:

$$\tilde{H}_n = \{e\} \leq \ldots \leq \tilde{H}_1 \leq \tilde{H}_0 = T_n$$

having the stated properties.

Now consider: $H_i' := H_i \cap G \subseteq G$ and the sequence

$$H_n' = \{e\} \subseteq H_{n-1}' \subseteq \dots \subseteq \tilde{H}_2' \subseteq \tilde{H}_1' \subset H_0' = G.$$

Then all $H_i' \subseteq G$ are normal subgroups. Moreover for $j: H_i' \hookrightarrow \tilde{H}_i$ we have $j^{-1}(\tilde{H}_{i+1}) = H_i' \cap \tilde{H}_{i+1} = H_{i+1}'$.

Thus we get an ~~of~~ injective morphism of groups

$$j: H_i'/H_{i+1}' \hookrightarrow \tilde{H}_i/\tilde{H}_{i+1}$$

$\Rightarrow H_i'/H_{i+1}'$ is a closed subgroup of either $\mathbb{G}_a$ or $\mathbb{G}_m$

$\Rightarrow H_i'/H_{i+1}'$ is either $\mathbb{G}_a$, $\mathbb{G}_m$ or a finite group.

To remove the possibility of finite groups as quotients, we have to modify the sequence. So consider now $H_i = H_i'^0$ and

$$H_n = \{e\} \subseteq \dots \subseteq H_2 \subseteq H_1 \subset H_0 = G.$$

Claim 1: Each H_i is normal in G.

As H_i' is normal, we get a morphism of affine varieties

$$c: G \times H_i' \longrightarrow H_i', \quad (g,h) \mapsto ghg^{-1}.$$

As $G \times H_i$ is connected, $c(G \times H_i)$ lies in one connected component. As $e \in c(G \times H_i)$ we get:

$$c|_{G \times H_i}: G \times H_i \longrightarrow H_i'^0 = H_i$$

$\Rightarrow H_i$ is normal.

Claim 2: Each H_i/H_{i+1} is either $\mathbb{G}_a$, $\mathbb{G}_m$ or the trivial group $\{e\}$.

$H_i'/H_{i+1} \longrightarrow H_i'/H_{i+1}'$ is a morphism of groups with finite kernel H_{i+1}'/H_{i+1}. Hence $\dim(H_i'/H_{i+1}) = \dim(H_i'/H_{i+1}') \leq 1$.

As we have a closed ~~immersion~~ embedding $H_i/H_{i+1} \hookrightarrow H_i'/H_{i+1}$ we get

$$\dim(H_i/H_{i+1}) \leq 1.$$

Moreover H_i/H_{i+1} is connected, as the image of the connected group H_i under $H_i \longrightarrow H_i/H_{i+1}$.

But every ~~trivial~~ connected group of dimension 0 is the trivial group.

And by E32) every connected group of dimension 1 is either G_a or G_m.

Claim 1 and 2 imply, that we get the desired sequence if we discard in

$$H_n \subseteq H_{n-1} \subseteq \dots \subseteq H_2 \subseteq H_1 \subseteq H_0 = G$$

all H_i with $H_i = H_{i-1}$ (i.e. all instances where the quotients H_i/H_{i-1} are the trivial group).

E32)

a) Let G act on itself by conjugation. Then the orbits are connected (because G is) and locally closed. In an irreducible space of dimension 1, such subsets are either single points or open dense subsets.
If there is an open dense orbit (i.e. conjugacy class) then the complement consists of finitely many points.
$\Rightarrow$ There are only finitely many conjugacy classes.
Otherwise all orbits consist of just one single point
$\Rightarrow$ G is abelian.

b) Consider only the case where G has finitely many conjugacy classes. Then there is an open conjugacy class $U \subseteq G$. Embed now $G \hookrightarrow GL_n$ and consider:

$$\alpha_i : GL_n \longrightarrow K, \quad A \longmapsto \left\{ \begin{array}{l} i\text{th coefficient of the char.} \\ \text{polynomial of } A \end{array} \right\}$$

This is a morphism of affine varieties, because the coeff. of the char. polynomial are linear combinations of determinants of minors of A, hence given by polynomials.
As the char. poly. is constant on conjugacy classes

$\alpha_i|_U : U \longrightarrow K$ is constant.

As $U \subseteq G$ is dense, $\alpha_i|_G : G \longrightarrow K$ is constant as well.

$\Rightarrow$ char. poly. is constant on all of G.

$\Rightarrow \forall g \in G: \quad \chi(g) = \chi(e) = (x-1)^n$

$\Rightarrow \forall g \in G: \quad g$ is unipotent.

c) In both cases G is solvable. Hence by E31) we have a normal series with filtration steps $\mathbb{G}_a$ or $\mathbb{G}_m$. With $\dim G = 1$ we can only have one such filtration step and

$$G \cong \mathbb{G}_a \quad \text{or} \quad G \cong \mathbb{G}_m.$$

Problem:

We have a circular argumentation here E31) $\rightleftarrows$ E32) !

Unfortunately there seems to be no easy way to break this circle. We nevertheless sketch two ways to remedy the situation, both a bit beyond your current knowledge.

Fix for E31) Claim 2:

It is not (too) hard to see that $H_i/H_{i+1} \longrightarrow H_i'/H_{i+1}$ is a finite étale morphism (the alg. geometric analogue of a topological covering map). Now one can show that if $X \longrightarrow K$ is a finite étale map and X connected, then X is ~~étale~~ isomorphic to K. Similarly if $X \longrightarrow K \setminus \{0\}$ is finite étale, X connected, then $X \cong K \setminus \{0\}$ (even though the map needs ~~to~~ not too be an isomorphism in both cases!).

$\Rightarrow H_i'/H_{i+1}$ is an algebraic group with underlying affine variety K or $K \setminus \{0\}$.

But a direct calculation shows, that there is only one possible group structure on K resp. $K \setminus \{0\}$, namely $\mathbb{G}_m$ or $\mathbb{G}_a$.

$\Rightarrow H_i'/H_{i+1}$ is isomorphic to $\mathbb{G}_m$ or $\mathbb{G}_a$

$\Rightarrow H_i/H_{i+1} = \mathbb{G}_m$ or $\mathbb{G}_a$ or $\{e\}$ with argumentation as before.

Fix for E 32, c):

Assume G is not abelian. Then by a) and b) G is unipotent and has an open conjugacy class. G unipotent implies that $\dim Z(G) \geq 1$, hence $Z(G) \subseteq G$ is open and of finite index.

$\Rightarrow$ All conjugacy classes are finite. Contradiction to "open conj. class"!

Thus G is abelian and by the classification of abelian alg. groups

$G = G_s$ or $G = G_u$.

If $G = G_s$, then G is diagonalizable (using G connected), hence a torus of dimension 1 $\Rightarrow$ $G \cong \mathbb{G}_m$.

If $G = G_u$, the situation is more difficult, cf. for example Springer: Linear algebraic groups, chapter 3.4.