Solutions to sheet 9

E33) There exists an embedding $G \hookrightarrow GL_n$, i.e. we may choose a faithful representation $\varrho: G \hookrightarrow GL(V)$ for some vector space V.

By assumption there is a G-invariant vector $v_1 \in V$. Then $V/\langle v_1\rangle$ is again a G-representation and we may choose a G-invariant vector $\bar v_2 \in V/\langle v_1\rangle$, which we lift to a vector $v_2 \in V$. Now $V/\langle v_1, v_2\rangle$ is a G-representation and we may choose $v_3 \in V$ s.th. $\bar v_3 \in V/\langle v_1, v_2\rangle$ is G-invariant...

In the end we obtain a basis $\{v_1, \ldots, v_n\}$ of V s.th. each v_i is G-invariant in $V/\langle v_1, \ldots, v_{i-1}\rangle$. Identify now $GL(V) \cong GL_n$ via this basis. Then the G-invariance assertion translates into

$$\begin{array}{ccc} G & \hookrightarrow & U_n \\ \varrho \downarrow & & \cap \\ GL(V) & \cong & GL_n \end{array}.$$

$\Rightarrow G$ is unipotent.

E34)

As G has a positive dimensional center and any subgroup of the center is normal, it suffices to show the existence of a ~~normal~~ subgroup in $Z(G)$ isomorphic to G_a. It even suffices to do so in the identity component $Z(G)^\circ$.

Now E31) implies that there is a (normal) (1-dimensional) subgroup of $Z(G)^\circ$ either isomorphic to G_a or G_m.

As G consists of unipotent elements, it cannot have a subgroup isomorphic to G_m. Hence we have found the desired subgroup of $Z(G)^\circ$ and hence of G.

Remark:

We cannot apply E31) directly to G because of the missing connectedness assumption. If we apply it to the identity component G^0, then we have the problem to ensure that the subgroup is normal in <u>all of</u> G.

E 35)

a) The multiplication is given by polynomials, hence a morphism of varieties. The inverse is given by

$$(x,y)^{-1} = (-x, -y \cancel{2x^3}).$$

and hence a morphism of varieties.

Commutativity follows from

$$(x,y)\cdot(x',y') = (x+x', y+y' + x^2x' + xx'^2) =$$
$$= (x'+x, y'+y + x'^2x + x'x^2) = (x',y')\cdot(x,y).$$

b) A direct computation shows that we can embed

$$G \hookrightarrow U_4$$
$$(x,y) \longmapsto \begin{pmatrix} 1 & x & x^2 & y \\ & 1 & 2x & x^2 \\ & & 1 & x \\ 0 & & & 1 \end{pmatrix}$$

Hence G is unipotent.

c) Consider any morphism of alg. groups.

$$(f,g): G_a \longrightarrow G$$

Then f and g satisfy $\forall x, x' \in K$

$$f(x+x') = f(x) + f(x') \qquad (1)$$
$$g(x+x') = g(x) + g(x') + f(x)^2 f(x') + f(x) f(x')^2. \qquad (2)$$

(1) implies that f defines an endomorphism of G_a and thus has the form

$$f(x) = \lambda_0 x + \sum_{i\geq 1} \lambda_i x^{(3^i)}$$

(cf. some exercise class).

Now choose $n \geq 0$ minimal s.th. $\lambda_n \neq 0$. Moreover write $g(x) = \sum \mu_i x^i$. Then (2) gives:

$$g(x+x') = g(x) + g(x') + \lambda_n^3 x^{2\cdot 3^n} x'^{3^n} + \lambda_n x^{3^n} x'^{2\cdot 3^n} + \text{higher order terms}.$$

Comparing coefficients in front of terms of total degree 3^{n+1} gives:

$$\mu_{3^{n+1}}(x+x')^{3^{n+1}} = \mu_{3^{n+1}}x^{3^{n+1}} + \mu_{3^{n+1}}x'^{3^{n+1}} + \lambda_n^3 x^{2\cdot 3^n} x'^{3^n} + \lambda_n^3 x^{2\cdot 3^n} x'^{3^n}$$

But in char 3 we have $(x+x')^{3^{n+1}} = x^{3^{n+1}} + x'^{3^{n+1}}$

$\Rightarrow \lambda_n^3 = 0 \quad \Rightarrow \lambda_n = 0$ Contradiction!

$\Longrightarrow f \equiv 0$

Now any morphism $\mathbb{G}_a^{\oplus 2} \longrightarrow G$ is a product of two morphisms of the form $\mathbb{G}_a \to G$, hence it maps as well constantly to zero on the first component of G. Thus it cannot be surjective

$\Rightarrow \mathbb{G}_a^{\oplus 2} \not\cong G$.

E36)

Consider any orbit $G(x_0)$ (for some $x_0 \in X$). Then we have to show $G(x_0) = \overline{G(x_0)}$ ← this here is the closure of the orbit.

But it does not matter whether we take the closure of $G(x_0)$ inside X or in the closed subvariety $\overline{G(x_0)}$. Hence we may replace X by $\overline{G(x_0)}$ and assume wlog that $G(x_0)$ is dense (and therefore automatically open) in X.

Now consider the complement $Z = \overline{G(x_0)} \setminus G(x_0)$, which is closed in $X = \overline{G(x_0)}$ and G-invariant. Let $\mathfrak{I}(Z) \subseteq K[X]$ be the ideal of functions vanishing on Z. Let $f \in K[X]$ (viewed as a function $f: X \to K$) and $g \in G$. Then by definition:

$$g(f): X \longrightarrow K, \quad g(f)(x) = f(g(x)).$$

$\Longrightarrow$ f vanishes on Z if and only if $g(f)$ vanishes on Z

$\Rightarrow \mathfrak{I}(Z)$ is a G-representation.

Assume now that $\mathfrak{I}(Z)$ is not the zero representation. As G is unipotent, there is a non-zero G-invariant vector $f \in \mathfrak{I}(Z)$. By definition of the G-action on $K[X]$, f has to be constant on G-orbits.

$\Rightarrow f$ is constant on $G(x_0)$

$\Rightarrow f$ is constant on $X = \overline{G(x_0)}$

$\Rightarrow f$ has no zero on all of X

$\Rightarrow Z = \emptyset$ and $G(x_0) = \overline{G(x_0)} = X$.

In the remaining case where $\mathfrak{I}(Z)$ is the zero representation, one immediately gets $Z = \emptyset$ from $\mathfrak{I}(Z) = (0)$.

$\Longrightarrow G(x_0) = \overline{G(x_0)}$ is closed.

Alternative solution to E34:

Embed $G \hookrightarrow U_n$. Applying E31, gives

$H_m = \{e\} \subseteq H_{m-1} \subseteq \dots \subseteq H_1 \subseteq H_0 = G$

with H_i normal in G and H_i / H_{i+1} either finite or $\mathbb{G}_m$ or $\mathbb{G}_a$.

~~As G is an~~ Let i be the maximal index with

$\dim H_i \geq 1$ (or equivalently $\dim H_i = 1$)

H_i is not necessarily connected, so consider H_i^0 the identity component. Then:

- H_i normal in G $\Rightarrow$ H_i^0 normal in G
- H_i^0 connected 1-dim. group and subgroup of a unipotent group $\Rightarrow H_i^0 \cong \mathbb{G}_a$