Technische Universität München
Zentrum Mathematik

WS 2014/15
Sheet 1

Prof. Dr. Gregor Kemper
M. Sc. Stephan Neupert

Linear Algebraic Groups (MA 5113)

Exercises (to be turned in: Wednesday, 15.10.2014, during the lecture)

In all exercises K denotes an algebraically closed field.

E 1 (Affine varieties)

(a) Show that $X_1 = \{(0,a) \mid a \in K\} \cup \{(1,0)\} \subset K^2$ is an affine variety.

(b) Show that the closed subsets of the affine variety $X_2 = V(x-y^2) \subset K^2$ are exactly X_2 itself and the finite subsets of X_2.

(c) For $X_3 = V(xy-1) \subset K^2$ let $f : K \to X_3$ be any morphism of affine varieties. Show that the image of f consists of a single point.

(d) Describe all irreducible components of X_1 and X_2.

E 2 (Normalizers and Centralizers in $\mathbf{GL_2}$)

(a) Show that for any element $h \in GL_2$ the map $c_h : GL_2 \to GL_2$, $c_h(g) = g \cdot h \cdot g^{-1}$ is a morphism of affine varieties.

(b) Let $h \in GL_2$ be some element. Show that the centralizer

$$\mathrm{C}_{GL_2}(h) = \{g \in GL_2 \mid g \cdot h \cdot g^{-1} = h\}$$

is a closed subgroup (and thus algebraic).

(c) Let $H \subset GL_2$ be any closed subgroup. Show that its normalizer

$$\mathrm{N}_{GL_2}(H) = \{g \in GL_2 \mid g \cdot h \cdot g^{-1} \in H \; \forall h \in H\}$$

is a closed subgroup.

Remark: Note that the definition of the normalizer given in the lecture uses a slightly stronger condition, which is nevertheless equivalent to the one above. Moreover all results will be generalized to arbitrary algebraic groups (instead of GL_2) later on.

E 3 (Different group structures on the same variety) Consider the two subgroups of GL_2 defined by

$$G_1 = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| \, a \in K^\times, b \in K \right\} \quad \text{and} \quad G_2 = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \middle| \, a \in K^\times, b \in K \right\}$$

Show that G_1 and G_2 are not isomorphic as algebraic groups. Prove that they are nevertheless isomorphic as affine varieties, once the group structure is forgotten.

E 4 (Finite groups are algebraic) Show that every finite group G can be seen as an algebraic group, i.e. that there is an injection (of sets) $G \hookrightarrow K^n$ (for some n) such that the image is an affine variety and both the multiplication map and the inverse are morphisms of affine varieties.

Solutions to the exercises will be available from October 16, 2014 on, at
`https://www-m11.ma.tum.de/lehre/wintersemester-201415/ws1415-linear-algebraic-groups/`
.