Technische Universität München WS 2014/15
Zentrum Mathematik Sheet 10

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Linear Algebraic Groups (MA 5113)

Exercises (to be turned in: Wednesday, 17.12.2014, during the lecture)

In all exercises K denotes an algebraically closed field and all algebraic groups are defined over this field.

E 37 (Characters of finite groups)

(a) Let $d \geq 1$ be some integer. Assume that $\operatorname{char} K = 0$ or that $\operatorname{char} K = p \nmid d$. Compute the group $X(\mathbb{Z}/d\mathbb{Z})$ for the finite group $\mathbb{Z}/d\mathbb{Z}$.

(b) Assume that $\operatorname{char} K = p$. Compute for any integer $n \geq 1$ the group $X(\mathbb{Z}/p^n\mathbb{Z})$.

E 38 (A perfect pairing) Define the group of cocharacters as $Y(D_n) = \operatorname{Hom}(\mathbb{G}_m, D_n)$.

(a) Give a definition of a natural group structure on $Y(D_n)$. Usually this group structure is written additively.

(b) Show that there is an isomorphism of groups $Y(D_n) \cong \mathbb{Z}^n$.

(c) Consider now the bilinear map

$$\langle -,- \rangle : X(D_n) \times Y(D_n) \to \operatorname{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z} \quad , \quad (\chi, \eta) \mapsto \chi \circ \eta$$

Show that this defines a perfect pairing, i.e. $\langle -,- \rangle$ induces isomorphisms

$$X(D_n) \to Y(D_n)^* = \operatorname{Hom}(Y(D_n), \mathbb{Z}) \quad , \quad \chi \mapsto \langle \chi, - \rangle$$

and

$$Y(D_n) \to X(D_n)^* = \operatorname{Hom}(X(D_n), \mathbb{Z}) \quad , \quad \eta \mapsto \langle -, \eta \rangle$$

E 39 (Inverses of cocharacters) Assume $\operatorname{char} K = 0$.

(a) Let $\eta \in Y(D_n)$ be an element given by an injective homomorphism $\eta : \mathbb{G}_m \to D_n$. Show that there is no positive integer $d \geq 2$ and no element $\eta' \in Y(D_n)$ satisfying $\eta = d \cdot \eta' \in Y(D_n)$ (using the additive notation for the group structure on $Y(D_n)$).

(b) Let $\eta \in Y(D_n)$. Show that $\eta : \mathbb{G}_m \to D_n$ is injective if and only if there is a character $\chi : D_n \to \mathbb{G}_m$ with $\chi \circ \eta = \operatorname{id}_{\mathbb{G}_m}$.

Hint: For the second part, show that injectivity of η implies the existence of a complement of $\mathbb{Z} \cdot \eta$ in $Y(D_n)$. Then try to apply exercise E38.
Remark: In positive characteristic, not every injective morphism has any of the other properties mentioned in this exercise, e.g. $\eta : \mathbb{G}_m \to \mathbb{G}_m$, $\eta(x) = x^p$ in characteristic p. Nevertheless all statements still hold, if one replaces "η injective" by a slightly stronger condition called "η is a closed immersion".

Please turn this page around for the last exercise.

E 40 (Character groups of non-diagonalizable groups) Let G be any algebraic group and $X(G) = \mathrm{Hom}(G, \mathbb{G}_m)$ the group of all characters. Define

$$H = \bigcap_{\chi \in X(G)} \ker \chi \subset G$$

(a) Show that $H \subset G$ is a closed normal subgroup.

(b) Show that the quotient G/H is diagonalizable.

(c) Prove $X(G) \cong X(G/H)$ and that $X(G)$ is a finitely generated group.

Solutions to the exercises will be available from December 18, 2014 on, at

`https://www-m11.ma.tum.de/lehre/wintersemester-201415/ws1415-linear-algebraic-groups/`