Technische Universität München
Zentrum Mathematik

WS 2014/15
Sheet 11

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Linear Algebraic Groups (MA 5113)

Exercises (to be turned in: Wednesday, 7.1.2015, during the lecture)

In all exercises K denotes an algebraically closed field and all algebraic groups are defined over this field.

E 41 **(Representations of tori)** Let G be a torus and $\rho : G \to GL(V)$ a finite dimensional G-representation. For every character $\chi \in X(G)$ define

$$V[\chi] = \{v \in V \mid \rho(g)(v) = \chi(g) \cdot v \ \ \forall g \in G\}$$

Show that $V \cong \oplus_{\chi \in X(G)} V[\chi]$ is a direct sum decomposition.

E 42 **(Complements of tori)** Assume that $\operatorname{char} K = 0$.

(a) Let $\mathbb{G}_m \subset D_n$ be a closed subgroup. Show that there exists a closed subgroup $H' \subset D_n$ such that the multiplication morphism $\mathbb{G}_m \times H' \to D_n$ is an isomorphism of algebraic groups. *Hint:* Use E39 to define H' as the kernel of a suitably chosen character.

(b) Let G be a diagonalizable algebraic group and $H \subset G$ a subgroup, which is a torus. Show that there exists a closed subgroup $H' \subset G$ such that the multiplication morphism $H \times H' \to G$ is an isomorphism of algebraic groups.

Remark: The assertions hold as well in positive characteristic.

E 43 **(Closed conjugacy classes)**

(a) Show that the conjugacy class of an element $g \in GL_n$ is closed if and only if g is semi-simple.

(b) Show that any conjugacy class in a unipotent group is closed.

E 44 **(Lie algebras of centralizers)** Assume $\operatorname{char} K \neq 2$. Consider the orthogonal group $H = O_5$ as a closed subgroup of GL_5 and the diagonal matrix $s = \operatorname{diag}(1,1,1,-1,-1) \in GL_5$.

(a) Compute the centralizer $\mathrm{C}_H(s)$ and deduce from this the Lie algebra $\mathscr{L}(\mathrm{C}_H(s))$.

(b) Determine the Lie algebra $\mathscr{L}(H)$ and use this to compute the centralizer

$$\mathfrak{c}_{\mathscr{L}(H)}(s) = \{x \in \mathscr{L}(H) \mid \operatorname{Ad}(s)(x) = x\} \subset \mathscr{L}(H).$$

Remark: It was shown in the lecture that $\mathscr{L}(\mathrm{C}_H(s)) = \mathfrak{c}_{\mathscr{L}(H)}(s)$. But do not use this result in your solution, as this exercise is meant to illustrate the formula in one explicit example.

Solutions to the exercises will be available from January 8, 2015 on, at

https://www-m11.ma.tum.de/lehre/wintersemester-201415/ws1415-linear-algebraic-groups/