Technische Universität München
Zentrum Mathematik

WS 2014/15
Sheet 12

Prof. Dr. Gregor Kemper
M. Sc. Stephan Neupert

Linear Algebraic Groups (MA 5113)

Exercises (to be turned in: Wednesday, 14.1.2015, during the lecture)

In all exercises K denotes an algebraically closed field and all algebraic groups are defined over this field.

E 45 **(Product decomposition in a unipotent group)** Let $\lambda, \mu \in K \setminus \{0\}$ be two distinct elements and define $s = \operatorname{diag}(\lambda, \lambda, \mu) \in D_3 \subset GL_3$.

(a) Compute $\mathrm{C}_{U_3}(s) = \{g \in U_3 \mid gsg^{-1} = s\}$.

(b) Compute the image of U_3 under the morphism of varieties $\gamma_s : GL_3 \to GL_3$, $\gamma_s(g) = sgs^{-1}g^{-1}$.

(c) Show that the product morphism $m : \mathrm{C}_{U_3}(s) \times \gamma_s(U_3) \to U_3$, $m(g,h) = g \cdot h$ is bijective and compute for each element in U_3 its preimage.

Remark: This is a special case of theorem 8.4, which was shown in the lecture.

E 46 **(A counterexample)** Consider the action of $\mathbb{G}_a \cong U_2$ on T_2 given by conjugation $U_2 \times T_2 \to T_2$, $(g,x) \mapsto g \cdot x \cdot g^{-1}$.

(a) Show that the morphism $f : T_2 \to D_2$, $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ is surjective and constant on U_2-orbits in T_2.

(b) Determine the centralizer $\mathrm{C}_{T_2}(U_2)$ or equivalently the set of all elements in T_2, which are invariant under the U_2-action. Conclude that the restricted morphism $f : \mathrm{C}_{T_2}(U_2) \to D_2$ is no longer surjective.

Remark: This shows that corollary 18.4 from the lecture may fail if a non-diagonalizable group acts.

E 47 **(An example of a Weyl group)** Consider the algebraic group $PGL_n = GL_n/Z(GL_n)$, where $Z(GL_n) \subset GL_n$ denotes the center.

(a) Find a torus T in PGL_n of dimension $n-1$.

(b) Compute the Weyl group $\mathrm{W}(PGL_n) = \mathrm{N}_{PGL_n}(T)/T$, where $\mathrm{N}_{PGL_n}(T)$ denotes the normalizer of T in PGL_n.

E 48 **(Maximal tori in an orthogonal group)**

(a) Find a 1-dimensional torus in SO_3.

(b) Show that any two 1-dimensional tori in SO_3 are conjugated.
Hint: Prove that for the canonical action of SO_3 on K^3, such a torus will fix some line pointwise.

(c) Prove that there does not exist any 2-dimensional torus in SO_3.

Solutions to the exercises will be available from January 15, 2015 on, at

`https://www-m11.ma.tum.de/lehre/wintersemester-201415/ws1415-linear-algebraic-groups/`