Technische Universität München
Zentrum Mathematik

WS 2014/15
Sheet 13

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# Linear Algebraic Groups (MA 5113)

**Exercises** (to be turned in: Wednesday, 21.1.2015, during the lecture)

In all exercises $K$ denotes an algebraically closed field and all algebraic groups are defined over this field.

**E 49 (Some semi-direct products)**

(a) Show that the symmetric group $S_n$ is isomorphic to a semi-direct product $A_n \rtimes \mathbb{Z}/2\mathbb{Z}$, where $A_n$ denotes the alternating group.

(b) Prove that we can write $GL_n$ as a semi-direct product $SL_n \rtimes H$ for some closed subgroup $H \subset GL_n$. Show that for $n \geq 2$, the subgroup $H$ is not uniquely determined.

(c) Show that the cyclic group $\mathbb{Z}/4\mathbb{Z}$ is not isomorphic to a semi-direct product $\mathbb{Z}/2\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$.

*Remark:* Note that part (c) gives an example of a surjective group morphism $f : G \to H$, such that there is no group morphism $g : H \to G$ satisfying $g \circ f = \mathrm{id}_H$.

**E 50 (Some counterexamples)**

(a) Give an example of a connected algebraic group $G$ and a closed subgroup $H \subset G$ consisting of semi-simple elements, such that $H$ does not lie in any maximal torus of $G$.
*Hint:* Consider non-connected subgroups $H$.

(b) Give an example of a connected algebraic group $G$, a maximal torus $T \subset G$ and a closed subgroup $H \subset T$, such that $\mathrm{C}_G(H) \neq \mathrm{N}_G(H)$.

*Remark:* This shows that the solvability of $G$ is essential in theorem 9.6.

**E 51 (Extensions of tori)** Let $G$ be an algebraic group and $T \subset G$ a torus, which is normal in $G$. Assume that $G/T$ is again a closed torus. Prove that $G$ itself is a torus.

**E 52 (Centralizer of maximal tori)** Let $G$ be a connected solvable algebraic group and $T \subset G$ a maximal torus. Show that the centralizer $\mathrm{C}_G(T)$ is nilpotent.

**Solutions to the exercises will be available from January 22, 2015 on, at**

https://www-m11.ma.tum.de/lehre/wintersemester-201415/ws1415-linear-algebraic-groups/