Technische Universität München
Zentrum Mathematik

WS 2014/15
Sheet 2

Prof. Dr. Gregor Kemper
M. Sc. Stephan Neupert

Linear Algebraic Groups (MA 5113)

Exercises (to be turned in: Wednesday, 22.10.2014, during the lecture)

In all exercises K denotes an algebraically closed field and all algebraic groups are defined over this field.

E 5 **(Images of morphisms of affine varieties)** Consider the affine variety $X = V(xz - y) \subset K^3$ and the morphism

$$f : X \to K^2 \quad , \quad (x, y, z) \mapsto (x, y).$$

Show that the image of this morphism is not locally closed in K^2, i.e. an intersection of an open subset with a closed one. In particular it is neither open nor closed.
Hint: To understand this example better, either draw pictures (over the reals) or view X as a family of affine varieties (how do they look like?) varying along the x-axis.
Remark: Note the completely different behaviour when working with algebraic groups!

E 6 **(Finite normal subgroups)**

(a) Show in detail that an algebraic group G acts *morphically* on $X = G$ by conjugation.

(b) Show that any finite normal subgroup of a connected algebraic group G lies in the center of G.

(c) Give examples showing that we cannot omit the assumption "normal" or the assumption "connected" in part b) of this exercise.

E 7 **(Subgroups fixed by automorphisms)** Let G be an algebraic group and $f : G \to G$ an automorphism of algebraic groups. Assume that $H \subseteq G$ is a closed subgroup satisfying $f(H) \subseteq H$. Show that $f(H) = H$.
Hint: Consider the dimension of the identity components H^0 and $f(H)^0$.
Remark: This removes the ambiguity in the definition of a normalizer noted on the previous exercise sheet.

E 8 **(Normalizers which are not connected)** Consider $D_n \subset GL_n$ the subgroup of all diagonal matrices.

(a) Show that the normalizer $\mathrm{N}_{GL_n}(D_n)$ consists exactly of all monomial matrices, i.e. of all elements in GL_n that have exactly one non-zero coefficient in each column and each row.

(b) What is the identity component $\mathrm{N}_{GL_n}(D_n)^0$?

Solutions to the exercises will be available from October 23, 2014 on, at

`https://www-m11.ma.tum.de/lehre/wintersemester-201415/ws1415-linear-algebraic-groups/`
.