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Linear Algebraic Groups (MA 5113)

Exercises (to be turned in: Wednesday, 29.10.2014, during the lecture)

In all exercises K denotes an algebraically closed field and all algebraic groups are defined over this field.

E 9 (Partial order on the set of orbits)

(a) Let G be an algebraic group acting on an affine variety X. On the set of orbits, we define the relation $\prec$ via

$$G(x) \prec G(y) \quad \text{if} \quad G(x) \subseteq \overline{G(y)}$$

for any two orbits $G(x)$ and $G(y) \subset X$. Show that $\prec$ defines a partial order on the set of orbits.

(b) Consider now the group $G = \mathbb{G}_m \times \mathbb{G}_m$ acting on the affine variety $X = K^2$ via

$$G \times X \to X \quad , \quad ((\lambda,\mu),(x,y)) \mapsto (\lambda \cdot x, \mu^2 \cdot y) \qquad \forall\ \lambda,\mu \in K^\times, x,y \in K$$

(here we identify $\mathbb{G}_m \cong K^\times$ by mapping $\lambda \in K^\times$ to $(\lambda,\lambda^{-1}) \in V(xy-1) \subset K^2$).
Show that this action has precisely four orbits and describe the partial order $\prec$ explicitly in this example.

E 10 (Conjugation action of GL_3) Consider the algebraic group $G = GL_3$ and the affine variety of all 3×3-matrices $X = K^{3\times 3}$. Then G acts on X by conjugation $g(x) = g \cdot x \cdot g^{-1}$ for all $g \in G = GL_3$ and $x \in X = K^{3\times 3}$.

(a) Show that the set $Y \subset X$ consisting on all nilpotent matrices is a closed subset of X, which is stable under the G-action.

(b) Determine all G-orbits in Y and describe the partial order $\prec$ (cf. the previous exercise) on them.

E 11 ((Non)-extension of group actions) Give an example of an algebraic group G acting morphically on an affine variety $X \subset K^n$ (for some n), such that there exists no action of G on all of K^n coinciding with the given action when restricted to X.
Hint: Such examples already exist for finite groups G.

E 12 (The Lie algebra of $\mathbb{G}_m$)

(a) Let R be any K-algebra, $S \subset R$ a multiplicative subset and $S^{-1}R$ the localization of R in S. Consider the map

$$\mathrm{Der}_K(S^{-1}R, S^{-1}R) \to \mathrm{Der}_K(R, S^{-1}R) \quad , \quad \delta \mapsto \delta \circ i$$

where $i : R \to S^{-1}R$ is the canonical morphism. Show that this gives a well-defined bijection between the two sets.

(b) Identifying $K[\mathbb{G}_m] = K[x,x^{-1}]$, show that $\mathrm{Der}_K(K[\mathbb{G}_m], K[\mathbb{G}_m]) = \{p \cdot \frac{d}{dx} \mid p \in K[x,x^{-1}]\}$.

(c) Verify that $\mathrm{Lie}(\mathbb{G}_m) = \{\lambda x \frac{d}{dx} \mid \lambda \in K\} \subset \mathrm{Der}_K(K[\mathbb{G}_m], K[\mathbb{G}_m])$.

Solutions to the exercises will be available from October 30, 2014 on, at
`https://www-m11.ma.tum.de/lehre/wintersemester-201415/ws1415-linear-algebraic-groups/`
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