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Linear Algebraic Groups (MA 5113)

Exercises (to be turned in: Wednesday, 5.11.2014, during the lecture)

In all exercises K denotes an algebraically closed field and all algebraic groups are defined over this field.

E 13 (Derivations) Let R be any K-algebra and $\delta_1, \delta_2 \in \mathrm{Der}_K(R,R)$ two derivations.

(a) Prove that $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$ is again a derivation.

(b) Give an example, such that $\delta_1 \circ \delta_2$ is not a derivation.

Project 1 (Lie algebra of GL_n)

(a) Show that the space of point derivations can be described as

$$\mathrm{Der}_K(K[GL_n], K) = \left\{ \sum_{i,j} \lambda_{ij} \frac{\partial}{\partial x_{ij}} \mid \lambda_{ij} \in K \, \forall i,j = 1,\dots,n \right\} \cong K^{n\times n}.$$

Here $\frac{\partial}{\partial x_{ij}}$ maps an element $f \in K[GL_n]$ to $\frac{\partial f}{\partial x_{ij}}(e)$ where we evaluate at the unit element $e \in GL_n$.

(b) The multiplication map $m : GL_n \times GL_n \to GL_n$ defines a homomorphism of coordinate rings

$$m^* : K[GL_n] = K[x_{ij}][\det(x_{ij})^{-1}] \to K[GL_n \times GL_n] = K[y_{ij}, z_{ij}][\det(y_{ij})^{-1}, \det(z_{ij})^{-1}]$$

where i,j always range between 1 and n and $\det(\dots)$ denotes the determinant polynomial in the respective variables. What is $m^*(x_{ij})$ explicitly for each coordinate x_{ij}?

(c) If we have two point derivations $\delta_1, \delta_2 : K[GL_n] \to K$, we set

$$\delta_1 \otimes \delta_2 : K[GL_n \times GL_n] \to K \quad , \quad y_{ij} \mapsto \delta_1(y_{ij}) \text{ and } z_{ij} \mapsto \delta_1(z_{ij})$$

(recall that we view K as a $K[GL_n \times GL_n]$-algebra via evaluation at (e,e) and use this to define $\delta_1 \otimes \delta_2$ on all polynomials). Moreover we define their Lie bracket via

$$[\delta_1, \delta_2] : K[GL_n] \to K \quad , \quad [\delta_1, \delta_2](f) = (\delta_1 \otimes \delta_2)(m^*(f)) - (\delta_2 \otimes \delta_1)(m^*(f))$$

Prove now that under the identification $\mathrm{Der}_K(K[GL_n], K) = K^{n\times n}$ constructed in (a), the Lie bracket of point derivations corresponds to the Lie bracket of matrices

$$K^{n\times n} \times K^{n\times n} \to K^{n\times n} \quad , \quad (A,B) \mapsto [A,B] = AB - BA.$$

Remark: One can show (by a direct, but annoying computation), that the Lie bracket defined above, coincides with the Lie bracket on $\mathscr{L}(GL_n)$ under the isomorphism $\mathrm{Der}_K(K[GL_n], K[GL_n])^{GL_n} \cong \mathrm{Der}_K(K[GL_n], K)$. Thus we can conclude that $\mathscr{L}(GL_n) \cong K^{n\times n}$ as Lie algebras.
Note moreover that all constructions done in this project can be generalized to arbitrary algebraic groups G.

Solutions to this project will give additional marks.

Please turn this page around for the other three exercises.

E 14 (Lie algebra of SL_n)

(a) Show that the canonical inclusion $i : SL_n \to GL_n$ induces an injection

$$d_e i : T(SL_n) \to T(GL_n) \cong K^{n\times n}.$$

(b) Describe the image of $d_e i$ explicitly and conclude that we have an isomorphism of Lie-algebras

$$\mathscr{L}(SL_n) \cong \{A \in K^{n\times n} \mid \operatorname{tr}(A) = 0\}$$

Here we use again the Lie bracket $[A,B] = AB - BA$ for all $A, B \in K^{n\times n}$ of trace 0.

E 15 (Derivations of some morphisms) Throughout this exercise we identify $T(GL_n) = K^{n\times n}$.

(a) Let $i_1 : GL_n \to GL_n \times GL_n$, $g \mapsto (g,1)$ and $i_2 : GL_n \to GL_n \times GL_n$, $g \mapsto (1,g)$ be the canonical inclusions. Show that their derivations yield an isomorphism

$$d_e i_1 \oplus d_e i_2 : T(GL_n) \oplus T(GL_n) \to T(GL_n \times GL_n)$$

(b) Using the identification of (a), show that the derivation of the multiplication map $m : GL_n \times GL_n \to GL_n$ is given by

$$d_e m : T(GL_n) \oplus T(GL_n) \cong T(GL_n \times GL_n) \to T(GL_n) \quad , \quad (A,B) \to A + B$$

Hint: Compute (in two ways!) the derivations of $m \circ i_1$ and $m \circ i_2$.

(c) Show that the derivation of the inverse map $\iota : GL_n \to GL_n$ is given by

$$d_e \iota : T(GL_n) \to T(GL_n) \quad , \quad A \to -A$$

Hint: Use that the composition $GL_n \xrightarrow{diagonal} GL_n \times GL_n \xrightarrow{(\mathrm{id},\iota)} GL_n \times GL_n \xrightarrow{m} GL_n$ maps everything to the unit element, hence has vanishing derivation.

(d) Show that the derivation of the transpose ${}^T : GL_n \to GL_n$ is given by

$$d_e{}^T : T(GL_n) \to T(GL_n) \quad , \quad A \to A^T$$

E 16 (Lie algebra of O_n) Consider the orthogonal group $O_n = \{A \in GL_n \mid A \cdot A^T = 1_n\} \subset GL_n$ and assume $char(K) \neq 2$. Use the statements of the previous exercise to show that its Lie algebra is given by

$$\mathscr{L}(O_n) = \{A \in K^{n\times n} \mid A = -A^T\}$$

as a sub-Lie algebra of $\mathscr{L}(GL_n) = K^{n\times n}$.
Remark: You may use the fact $\dim O_n = \frac{1}{2}n(n-1)$ without proof.

Solutions to the exercises will be available from November 6, 2014 on, at

`https://www-m11.ma.tum.de/lehre/wintersemester-201415/ws1415-linear-algebraic-groups/`

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