Technische Universität München
Zentrum Mathematik

WS 2014/15
Sheet 6

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# Linear Algebraic Groups (MA 5113)

**Exercises** (to be turned in: Wednesday, 19.11.2014, during the lecture)

In all exercises $K$ denotes an algebraically closed field and all algebraic groups are defined over this field.

**E 21** **(Jordan decomposition)** Give an example of a (non-closed) subgroup $G \subset GL_n$ and an element $g \in G$ with Jordan decomposition $g = g_s \cdot g_u$ with $g_s, g_u \in GL_n$ such that $g_s, g_u \notin G$.

**E 22** **(Connectedness of unipotent groups)** For this exercise assume $\operatorname{char} K = 0$.

(a) Let $G$ be an arbitrary algebraic group. Show that any element of finite order in $G$ is semi-simple.

(b) Let $G \subset GL_n$ be a subgroup consisting of unipotent elements. Show that $G$ is connected.

*Remark:* Note that these statements are wrong in positive characteristic.

**E 23** **(Commutative groups)** Let $G$ be an algebraic group. Show that the maps

$$\varphi_s : G \to G \quad , \quad g \mapsto g_s$$
$$\varphi_u : G \to G \quad , \quad g \mapsto g_u$$

are morphisms of affine varieties. Deduce for a commutative algebraic group $G$, that the morphism $(\varphi_s, \varphi_u) : G \to G_s \times G_u$ is a morphism of algebraic groups.
*Remark:* This finishes the proof of $G \cong G_s \times G_u$ for commutative algebraic groups.

**E 24** **(Non-closed commutator subgroups)** For any two closed subgroups $A, B \subset GL_n$, define their commutator subgroup $(A, B) \subset GL_n$ as the subgroup generated by all elements of the form $aba^{-1}b^{-1}$, $a \in A$, $b \in B$. Give an example such that the commutator subgroup $(A, B) \subset GL_n$ is not closed.
*Hint:* Use for $A$ and $B$ finite subgroups generated by non-commuting elements.

**Solutions to the exercises will be available from November 20, 2014 on, at**

https://www-m11.ma.tum.de/lehre/wintersemester-201415/ws1415-linear-algebraic-groups/
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