Technische Universität München
Zentrum Mathematik

WS 2014/15
Sheet 7

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Linear Algebraic Groups (MA 5113)

Exercises (to be turned in: Wednesday, 26.11.2014, during the lecture)

In all exercises K denotes an algebraically closed field and all algebraic groups are defined over this field.

E 25 (Commutator subgroups) Let G be any group.

(a) Let N, N' be two normal subgroups of G. Show that the commutator subgroup (N, N') is again normal in G.

(b) Let $H \subseteq G$ be any subgroup and $\mathrm{N}_G(H) = \{g \in G \,|\, gHg^{-1} = H\}$ its normalizer. Show that $\mathrm{N}_G(H) \subseteq G$ is the largest subgroup $N \subseteq G$ satisfying $(H, N) \subseteq H$.

(c) Let $H \subseteq G$ be any subgroup. Show $\mathrm{N}_G(H) \subseteq \mathrm{N}_G((H,H))$ and give an example where equality does not hold.

(d) Let $f: G \to G'$ be a homomorphism between two groups. Show that $f(\mathcal{D}^i(G)) \subseteq \mathcal{D}^i(G')$ and $f(\mathcal{C}^i(G)) \subseteq \mathcal{C}^i(G')$ for each $i \geq 1$.

(e) Assume that the homomorphism $f: G \to G'$ is surjective. Prove now $f(\mathcal{D}^i(G)) = \mathcal{D}^i(G')$ and $f(\mathcal{C}^i(G)) = \mathcal{C}^i(G')$ for each $i \geq 1$.

E 26 (Non-extension of nilpotent groups) Give an example of a group G and a normal subgroup $N \subset G$ such that

(i) N and G/N are nilpotent, but

(ii) G is not nilpotent.

E 27 (Ascending central series) Let G be a group. Then we define the *ascending central series* $Z_i(G)$ inductively via $Z_0(G) = \{e\}$ and for all $i \geq 0$

$$Z_{i+1}(G) = \pi_i^{-1}\left(Z(G/Z_i(G))\right) \quad \text{with} \quad \pi_i : G \to G/Z_i(G)$$

where $Z(G/Z_i(G))$ denotes the center of the group $G/Z_i(G)$.

(a) Show that $Z_i(G) \subset G$ is normal, which implies that $Z_{i+1}(G)$ is indeed well-defined.

(b) Assume that G is nilpotent. Show that there exists an integer $n \geq 1$ such that $Z_n(G) = G$.

(c) Prove the converse as well: If G is a group such that $Z_n(G) = G$ for some integer n, then G is nilpotent.

E 28 (Semi-simple elements in nilpotent groups) Let G be a nilpotent group and $g \in G_s$ a semi-simple element. Show that $\mathrm{Ad}(g) \in GL(\mathscr{L}(G))$ is the identity element.

Hint: Consider the morphism $f: G \to G$, $x \mapsto gxg^{-1}x^{-1}$. Use that some power of f is trivial to conclude that $\mathrm{Ad}(g)$ is unipotent.

Remark: If we assume in addition that G is connected, then it is even true that g lies in the center of G. However the proof of this fact requires far more work.

Solutions to the exercises will be available from November 27, 2014 on, at

`https://www-m11.ma.tum.de/lehre/wintersemester-201415/ws1415-linear-algebraic-groups/`

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