Technische Universität München
Zentrum Mathematik

WS 2014/15
Sheet 9

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Linear Algebraic Groups (MA 5113)

Exercises (to be turned in: Wednesday, 10.12.2014, during the lecture)

In all exercises K denotes an algebraically closed field and all algebraic groups are defined over this field.

E 33 **(Unipotent groups characterized by representations)** Let G be an algebraic group. Assume that for every representation $\rho : G \to GL(V)$ there exists a non-zero G-invariant vector $v \in V$. Show that G is unipotent.

E 34 **(Additive subgroups)** Let G be a unipotent algebraic group of dimension at least 1. Prove that G contains a normal subgroup isomorphic to $\mathbb{G}_a$.

E 35 **(Unipotent 2-dimensional groups)** Consider the algebraic group G with underlying affine variety K^2 and multiplication given by

$$(x,y) \cdot (x',y') = (x+x', y+y'+x^2 \cdot x' + x \cdot x'^2) \qquad \text{for all } (x,y),(x',y') \in K^2$$

(a) Show that G is an abelian algebraic group.

(b) Prove that G is unipotent.

(c) Assume now that K has characteristic $p = 3$. Show that G is not isomorphic to $\mathbb{G}_a^2$.

Hint: For the last part of this exercise, show that any morphism $f : \mathbb{G}_a \to G$ is constantly zero on the first coordinate of G.
Remark: Via a similar construction one may define, for every field K of positive characteristic, a 2-dimensional connected, abelian, unipotent group not isomorphic to $\mathbb{G}_a^2$. Note however that in characteristic 0 one may show that all 2-dimensional unipotent groups are isomorphic to $\mathbb{G}_a^2$ (and in particular automatically abelian).

E 36 **(Orbits of unipotent groups)** Let G be a unipotent algebraic group acting on an affine variety X. Show that each orbit of G in X is closed.
Hint: Reduce first to the situation that the orbit is open in X. Then consider the ideal in $K[X]$, defining its complement. What can you say about G-invariant elements in this ideal?

Solutions to the exercises will be available from December 11, 2014 on, at

`https://www-m11.ma.tum.de/lehre/wintersemester-201415/ws1415-linear-algebraic-groups/`
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