Integral canonical models of Shimura varieties and the Tate conjecture for K3s in positive characteristic

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Madapusi Pera [Mad13] gave a proof of the Tate conjecture for K3 surfaces in (odd) positive characteristic, namely that for any K3 surface X over a finitely generated field k of characteristic $p > 2$ and for any prime $\ell \neq p$, the ℓ-adic Chern class map defines an isomorphism

$$Pic(X) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H^2_{\acute{e}t}(X_{k^{sep}}, \mathbb{Q}_\ell(1))^{Gal(k^{sep}/k)}.$$

This was made possible by a recent breakthrough in the theory of canonical integral models of Shimura varieties (not necessarily of PEL-type) made by Kisin [Kis10] (using a similar approach as Vasiu, cf. [Vas99] and his other works). Note that the Tate conjecture is only one of many applications Kisin's work, see e.g. [Zha13] and [Kis11].
Given these developments, we first want to understand Kisin's main ideas how to use Kottwitz's integral models of Shimura varieties of PEL-type to extend this construction first to all Shimura varieties of Hodge type and then further to all of abelian type. In the second half we want to understand how such integral models of Shimura varieties can be helpful in understanding K3s by analyzing their role in the proof of the Tate conjecture given in [Mad13].

1 Shimura varieties over $\mathbb{C}$ (short talk)

We introduce Shimura varieties over $\mathbb{C}$ as the complex analytic space given by the double quotient

$$Sh_K(G, X)(\mathbb{C}) = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f)/K).$$

Here G is a reductive group defined over $\mathbb{Q}$, $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$, $X \subset \mathrm{Hom}_{\mathbb{R}}(\mathbb{S}, G_{\mathbb{R}})$ is a $G(\mathbb{R})$-conjugacy class (subject to some compatibility conditions) and $K \subset G(\mathbb{A}_f)$ is a compact subgroup. The aim of this talk is to give the precise definition of Shimura varieties over $\mathbb{C}$ and to summarize existence results of models over the reflex field E (which is a number field). An overview is given in [Moo98, section 1 and 2], but consult as well Deligne's original articles.
Finally an overview over the remaining talks is given and they are distributed among the participants of this seminar.

2 Integral models of PEL-type Shimura varieties

The goal of this talk is to present Milne's definition of a canonical integral model (cf. [Kis10, 2.3.7])[1] and review its construction in the PEL-case. A good reference is Kottwitz's paper [Kot91] (mainly sections 5 and 8), though there are many other references. Rather than presenting every detail (like the precise formulation of the determinant condition or representability results), the focus should lie on convincing the audience that this construction indeed has the correct fiber over $\mathbb{C}$ and satisfies Milne's extension property. If time is too short, the speaker may consider to check these properties only in the Siegel case.

3 Integral models of Shimura varieties of Hodge type I

Following Kisin's article [Kis10] we construct canonical integral models of Shimura varieties of Hodge type by embedding them into the ones of PEL-type. More precisely any embedding of Shimura data

[1] The class of test schemes seems to differ from paper to paper. See [Moo98, 3.5 and 3.9] for a discussion about this.

$i : (G, X) \to (GSp, S^{\pm})$ defines an inclusion of Shimura varieties $Sh_K(G, X) \to Sh_{K'}(GSp, S^{\pm})$ over the reflex field (for the precise and more general statement cf. e.g. [Kis10, 2.1]). We have already seen that $Sh_{K'}(GSp, S^{\pm})$ admits a canonical integral model $\mathscr{S}_{K'}(GSp, S^{\pm})$. Then we take the normalization of the closure of $Sh_K(G, X)$ in $\mathscr{S}_{K'}(GSp, S^{\pm})$ as our candidate of choice for the canonical integral model $\mathscr{S}_K(G, X)$ of $Sh_K(G, X)$.
The goal of this talk is to state [Kis10, theorem 2.3.8.] and prove part (2), i.e. to see that $\mathscr{S}_K(G, X)$ has all properties of a canonical integral model except smoothness. Of course the cited statements in the proof of loc. cit. should be elaborated.
In talks 3-6 we may always assume $p > 2$ (as we need this assumption later on anyway).

4 Integral models of Shimura varieties of Hodge type II (long talk!)

It remains to see that $\mathscr{S}_K(G, X)$ is smooth, which is in by far the hardest part. This means it requires some preparations (namely [Kis10, 1.3-1.5]), which are the topic of this talk. The bare minimum to be covered includes the following

- If $G \subset GL(M)$ is a faithful representation (over some DVR), then G can be described as the point-wise stabilizer of elements in certain tensor products of M with itself [Kis10, prop. 1.3.2].
- Properties of the functor from p-divisible groups to φ-modules given in [Kis10, corollary 1.4.3]. This functor is given by a version of Fontaine's functor on crystalline representations (cf. [Kis10, theorem 1.2.1]).
- Define the ring R_G and state [Kis10, corollary 1.5.11].

5 Integral models of Shimura varieties of Hodge type III

Introduce absolute Hodge cycles as given in [Kis10, 2.2] up to corollary 2.2.2 and then explain the key proposition 2.3.5 in detail, implying (almost) immediately the desired smoothness result.

6 Integral models of Shimura varieties of abelian type

For time reasons, we will only briefly treat the case of Shimura varieties of abelian type. The goal is to convince the audience to "believe" in the existence of canonical integral models [Kis10, theorem 3.4.10 and corollary 3.4.14] and to understand its construction given by equation (3.4.11) in loc. cit. The key insight seems to be the alternative description of the Shimura variety over the reflex field given in [Kis10, proposition 3.3.10].

7 The Kuga-Satake construction

Present the Kuga-Satake construction (cf. e.g. [Huy, §4]), which is a purely analytical method to associate to a K3 surface (together with an ample line bundle) an abelian variety of dimension 2^{19} (together with a polarization), or equivalently to associate a (polarized) Hodge structure of weight 1 to a (polarized) Hodge structure of K3-type (which has weight 2!).
Our next aim is to understand this construction in families (still over $\mathbb{C}$), which is by no means easy. Summing things up, the passing from K3s to abelian varieties is done by going through the diagram

$$\mathrm{M}_{2d} \xleftarrow{2:1} \widetilde{\mathrm{M}}_{2d} \xrightarrow{Torelli} Sh_{K_0}(L_d) \underset{\text{étale}}{\xleftarrow{finite}} Sh_K(L_d) \underset{Satake}{\xrightarrow{Kuga-}} Sh_{\mathcal{K}}(C(L_d)).$$

Here L_d is a specific quadratic lattice (cf. [Mad13, 2.10]), $C(L_d)$ is its Clifford algebra, M_{2d} is the moduli space of K3 surfaces (with level structure), $\widetilde{\mathrm{M}}_{2d}$ is a two-fold cover over it, $Sh_{K_0}(L_d)$[2] is the

[2]There is a conflict of notation between [Mad13] and [Mad12]! In [Mad12] Shimura data and Shimura varieties without an index 0 refer to groups $G = GSp(...)$ and an index 0 refers to groups $G = SO(...)$. On the other hand in [Mad13] only Shimura varieties associated to $G = SO(L_d)$ are considered and the index 0 is dropped. Here we will use the notation of [Mad12], but adding L_d to it.

Shimura variety associated to $SO(L_d)$, $Sh_K(L_d)$ is associated to $GSp(L_d)$ and $Sh_{\mathcal{K}}(C(L_d))$ is associated to $GSp(C(L_d))$. The relevant arguments can be found in [Mad12, 3.1 - 3.10] and [Mad13, proposition 3.3 and 4.2][3]. Nevertheless it is recommended to follow the excellent exposition [Riz10, section 2.5], even though the statements there are (marginally) weaker than in [Mad12].
A note for the further talks: We just defined a morphism $\widetilde{\mathrm{M}}_{2d} \times_{Sh_{K_0}(L_d)} Sh_K(L_d) \to Sh_{\mathcal{K}}(C(L_d))$ over $\mathbb{C}$, where the source is a finite (non-étale) cover of the moduli space of K3 surfaces (with level structure). In fact Galois-descent allows us to define this morphism over $\mathbb{Q}$ and then the extension property of canonical models extends this morphism even over $\mathbb{Z}[\frac{1}{2}]$. But once we use this universal property we lose control of its properties. This is the reason why we need all the deformation arguments given later in talks 10 and 11. Using them we can check the desired properties in characteristic 0 and then reduce everything mod p.

8 The Tate conjecture for K3 surfaces of finite height

Before going into the details of [Mad13], we want to see the basic approach to the Tate conjecture. Its hard direction is to show that every $Gal(k^{sep}/k)$-invariant element in $H^2_{ét}(X_{k^{sep}}, \mathbb{Q}_\ell(1))$ actually comes from an invertible sheaf. Now, if $(X, \mathscr{L})$ is a K3 surface over a finite field with an ample invertible sheaf $\mathscr{L}$, then one would expect that the Kuga-Satake abelian variety A associated to $(X, \mathscr{L})$ (which we have not even defined yet) should exist after a finite field extension and that it has the property that

$$P(X, \mathscr{L}) \subseteq \mathrm{End}(H^1_{ét}(A_{k^{sep}}, \mathbb{Q}_\ell)),$$

where $P(X, \mathscr{L})$ denotes primitive cohomology, that is, the orthogonal complement of $c_1(\mathscr{L})$ inside $H^2_{ét}(X_{k^{sep}}, \mathbb{Q}_\ell(1))$ with respect to the Poincaré duality pairing. Thus the Kuga-Satake construction translates the characterization and existence of invertible sheaves on X in terms of Galois-actions into the characterization and existence of endomorphisms on abelian varieties in terms of Galois-actions, and the latter is classically known. Now, even if the Kuga-Satake construction is not available in positive characteristic (or if its properties are completely mysterious), one could try to do the following detour via characteristic 0: find a "sufficiently good" lift of the pair $(X, \mathscr{L})$ to characteristic 0, apply the Kuga-Satake construction over $\mathbb{C}$ in a way that is compatible with lifted Galois-actions, and then reduce back to characteristic p.
For ordinary K3 surfaces, Nygaard [Nyg83] realized this strategy using *canonical lifts* of such surfaces. Later, Nygaard and Ogus [NO85] made it also work for K3 surfaces that are non-supersingular by introducing *quasi-canonical lifts* (they are not canonical, but good enough).

9 The canonical integral model of the Shimura variety of K3-type

We gave in talk 7 the Kuga-Satake-construction as a series of morphisms and lifts. In order to get a better understanding of Kuga-Satake in positive characteristic, we want to define each step on integral level. To do so we need in particular canonical integral models of $Sh_K(L_d)$ and $Sh_{K_0}(L_d)$, which we construct in this talk. Unfortunately these Shimura varieties are only close to the ones considered by Kisin, so the following construction is needed to obtain [Mad12, theorem 8.1]:
Embed $L_d \subset \widetilde{L}$ as quadratic lattices such that $\widetilde{L}$ is self-dual over $\mathbb{Z}_{(p)}$ (which can be done by [Mad12, lemma 6.8][4]). Then $G = GSp(\widetilde{L})$ defines a Shimura datum of Hodge type and $G_0 = SO(\widetilde{L})$ defines one of abelian type. Hence Kisin's results (cf. talks 3 and 6) apply[5] and give us integral models $\mathscr{S}_{\overline{K}}$ and $\mathscr{S}_{\overline{K}_0}$ over $\mathbb{Z}_{(p)}$ (actually the first one is a Galois-cover over the second one), cf. [Mad12, theorem 4.4]. Now we can define a finite unramified cover $Z_{K^p}(\Lambda)$ over $\mathscr{S}_{\overline{K}}$ ([Mad12, proposition 6.13]) such that the morphism $Sh_K(L_d) \to Sh_{\overline{K}}(\widetilde{L})$ lifts canonically to $Z_{K^p}(\Lambda)$ over the reflex field $\mathbb{Q}$ ([Mad12, 6.15]). Then define as in [Mad12, theorem 7.2] the canonical integral model of $Sh_K(L_d)$ as the closure of the image of $Sh_K(L_d)$ in $Z_{K^p}(\Lambda)$ (with additional complications arising in the case $t = 2$). The same approach works (almost verbatim) for $\mathscr{S}_{K_0}$ as well.

[3] or ask Stephan Neupert for a detailed explanation.

[4] The cited lemma shows it for arbitrary lattices. But the statement becomes trivial if one restricts to the lattices L_d, as those are *defined* as sublattices of the selfdual lattice $N^{\oplus 3} \oplus E_8^{\oplus 2}$, cf. [Mad13, 2.10].

[5] We only need selfduality of the lattice to ensure that the compact subgroups $\overline{K}_p$ are hyperspecial.

Note that we are only interested in very special lattices L_d, but the arguments in this talk work for every L of signature $(n, 2)$ for some $n \geq 1$.

10 Deformation of special endomorphisms

It is easy to see that special morphisms[6]can be lifted to characteristic 0 inside the moduli space $\mathscr{S}_{\mathcal{K}}(C(L_d))$ of 2^{19}-dimensional abelian varieties. But it is much harder to do this deformation inside the integral model $\mathscr{S}_{K_0}$ of $Sh_{K_0}(L_d)$! This talk is devoted to the proof of this fact [Mad12, corollary 8.11], which by [Mad12, proposition 5.16] can be reduced to a deformation problem of isotropic lines in a fixed vector space. This new deformation problem can be solved very explicitly as carried out in [Mad12, proposition 5.18]. Note that (most of) section 5 uses the Shimura variety of Hodge type $\mathscr{S}_{\overline{K}}$ associated to $GSp(\widetilde{L})$ (or any other self-dual lattice), and the result is carried over to $\mathscr{S}_{K_0}$ only afterwards.

11 $\widetilde{\mathrm{M}}_{2d} \to \mathscr{S}_{K_0}$ is étale

In the previous talk we have seen how to lift special morphisms to characteristic 0 inside $\mathscr{S}_{K_0}$[7]. But rather than deforming it in the Shimura variety, we want to deform it in the moduli space of K3-surfaces $\widetilde{\mathrm{M}}_{2d}$. Thus we devote this talk to study their relationship:
We first note ([Mad13, 4.1-4.7]) that the period morphism over $\mathbb{C}$ (called Torelli in talk 7) descends first to the reflex field $\mathbb{Q}$ and then (by Milne's extension property) to a morphism $\widetilde{\mathrm{M}}_{2d} \to \mathscr{S}_{K_0}$ over $\mathbb{Z}[\frac{1}{2}]$[8]. It seems advisable only to sketch these arguments. Then present (with proof) the key statement [Mad13, theorem 4.8] of this talk, namely that this morphism is in fact étale.

12 The Tate conjecture in odd characteristic

Using the results of the previous talks, we can now prove [Mad13, theorem 4.17], stating that the Kuga-Satake construction preserves important properties even in positive characteristic. Following this we may then finish the prove of the Tate conjecture in characteristic $p > 2$ as done in [Mad13, 5.1-5.12]. Compared to talk 8 new complications arise from the fact, that we need many different lifts to characteristic 0 (instead of just one *canonical* one) in order to realize all classes. In fact the Tate conjecture tells us, that the Picard rank of a supersingular K3 surface is 22, while any K3 over characteristic 0 has Picard rank at most 20. So we cannot even hope for realizing all classes with just one lift.

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[6]If they were not already introduced in talk 8, give the several alternative definitions [Mad12, 5.2, 5.3 and 5.8] and state their equivalence [Mad12, corollary 5.21].

[7]Recall that this Shimura variety is called $\mathscr{S}_K(L_d)$ or simply $\mathscr{S}_K$ in [Mad13].

[8]In all previous talks we worked over $\mathbb{Z}_{(p)}$ for any prime $p > 2$ (or generalizations of this to larger number fields). Either note that this actually implies existence of canonical integral models over $\mathbb{Z}[\frac{1}{2}]$ together with all desired properties or continue to use the base $\mathbb{Z}_{(p)}$ as we are only interested in the fibers over $\mathbb{F}_p$ and $\mathbb{Q}$ anyway.

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