Solutions to sheet 1

A1) (a) Yes, topology induced by euclidean metric
$$d(x,y) = |x-y|.$$
(b) No, $\bigcup_{n\in\mathbb{Z}} \{n\}$ is not open in $\mathcal{T}_2$.

(c) Yes, if $U_i \in \mathcal{T}_3$ ($i \in I$ some index set), then $\bigcup_{i\in I} U_i \in \mathcal{T}_3$ because $X \setminus \bigcup_{i\in I} U_i \subseteq X \setminus U_i$ finite and (if I finite) $\bigcap_{i\in I} U_i \in \mathcal{T}_3$, because $X \setminus \bigcap_{i\in I} U_i = \bigcup_{i\in I} (X \setminus U_i)$ finite.

(d) No, $((0,1)\times(0,1)) \cup ((2,3)\times(2,3))$ is not in $\mathcal{T}_4$.

A2) (a)
- $\emptyset, X \in \bigcap_i \mathcal{T}_i$, because they are in all of the $\mathcal{T}_i$.
- If $U_j \in \bigcap_i \mathcal{T}_i$ (for $j \in J$ some index set), then $\bigcup_{j\in J} U_j \in \mathcal{T}_i$ for all i, i.e. $\bigcup_{j\in J} U_j \in \bigcap_i \mathcal{T}_i$
- If $U_j \in \bigcap_i \mathcal{T}_i$ (for $j \in J$ some finite index set), then $\bigcap_{j\in J} U_j \in \mathcal{T}_i$ for all i, i.e. $\bigcap_{j\in J} U_j \in \bigcap_i \mathcal{T}_i$

(b) Let $X = \mathbb{R}$, $\mathcal{T}_1 = \{\emptyset, \{0\}, X\}$, $\mathcal{T}_2 = \{\emptyset, \{1\}, X\}$. Then $\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{0\}, \{1\}, X\}$ which is not a topology, because it misses $\{0,1\}$.

(c) Let $S = \{\mathcal{T}' \mid \mathcal{T}' \text{ topology on } X \text{ and } \mathcal{T}_1 \cup \mathcal{T}_2 \subseteq \mathcal{T}'\}$.
We claim that $\mathcal{T} = \bigcap_{\mathcal{T}'\in S} \mathcal{T}'$ satisfies all requirements.
- $\mathcal{T}$ is a topology by part (a).
- $\mathcal{T}$ contains $\mathcal{T}_1 \cup \mathcal{T}_2$ by construction
- If $\mathcal{T}'$ is any topology containing $\mathcal{T}_1 \cup \mathcal{T}_2$, then $\mathcal{T}' \in S$ and by construction $\mathcal{T} = \bigcap_{\mathcal{T}''\in S} \mathcal{T}'' \subseteq \mathcal{T}'$, i.e. $\mathcal{T}'$ finer than $\mathcal{T}$.

A3) (a) f is continuous, because $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\mathbb{R}) = \mathbb{R}$ and
$$f^{-1}((-r,r)) = \begin{cases} (-r,r) & \text{if } r < 1 \\ (-1,1) & \text{if } 1 \le r \le 2 \\ (-\frac{r}{2}, \frac{r}{2}) & \text{if } 2 < r \end{cases}$$

f is not open, because $f((-3,3)) = (-6,-2] \cup (-1,1) \cup [2,-6]$ is not open.

f is not closed, because $f(\mathbb{R}) = \mathbb{R} \setminus ((-2,-1] \cup [1,2))$ not closed.

(b) We claim that $\mathcal{B} = \{(-r,r) \mid r \in \mathbb{Q}, r > 0\} \subset \mathcal{T}$ is a basis.

Indeed for any $r' \in \mathbb{R}, r' > 0$:

$$(-r', r') = \bigcup_{\substack{0<r<r' \\ r \in \mathbb{Q}}} (-r, r)$$

Moreover $\mathcal{B}$ is countable.

(c) We claim that $x \in \mathbb{R}$ is a limit if and only if $|x| \geq 1$.

- If $|x| \geq 1$, then any open neighbourhood of x (except for $\mathbb{R}$ itself, where everything is trivial) is of the form $(-r, r)$ for $r > |x| \geq 1$. Then for all $n > (r-x)^{-1}$: $x_n \in (-r, r)$
  $\Rightarrow$ x is limit.
- If $|x| < 1$, choose some $r < 1$, $r > |x|$. Then $(-r, r)$ is a neighbourhood of x containing none of the x_i.
  $\Rightarrow$ x is not a limit.

(d) $(-r, r) \subseteq M$ if and only if $r \leq 1$. Thus

$$\mathring{M} = \bigcup_{r<1} (-r, r) = (-1, 1)$$

$\overline{M} = \mathbb{R}$, because no other closed set contains $0 \in M$.

$\partial M = \overline{M} \setminus \mathring{M} = \mathbb{R} \setminus (-1, 1)$.