Solutions to Sheet 10

A 27)

(a) Let $x \in X$ be a point and choose any path $\alpha : I \to X$ from x_0 to x. Then $p \circ f \circ \alpha = p \circ \alpha : I \to Y$, i.e. $f \circ \alpha : I \to X$ is a lift of $p \circ \alpha$ starting at $f(x_0) = f \circ \alpha(0)$.
By theorem 8.24. such a lift is unique. In particular it's endpoint is unique:

$$f \circ \alpha(1) = f(x)$$

Hence fixing $f(x_0)$, determines $f(x)$, i.e. all of f.

(b) We have already seen, that α' defines an isomorphism

$$\alpha'_* : \pi_1(X, x_0) \xrightarrow{\sim} \pi_1(X, x_1)$$
$$\gamma \longmapsto \alpha'^{-1} * \gamma * \alpha'$$

So applying p_* gives:

$$p_*(\pi_1(X, x_1)) = \{ p_*(\alpha'^{-1} * \gamma * \alpha') \mid \gamma \in \pi_1(X, x_0) \} =$$
$$= \{ p_*(\alpha'^{-1}) * p_*(\gamma) * p_*(\alpha') \mid \gamma \in \pi_1(X, x_0) \} =$$
$$= \{ [\alpha^{-1}] * p_*(\gamma) * [\alpha] \mid \gamma \in \pi_1(X, x_0) \} =$$
$$= [\alpha^{-1}] * \cancel{p_*} \, p_*(\pi_1(X, x_0)) * [\alpha].$$

(c) Assume first that $p_*(\pi_1(X, x_0))$ is normal in $\pi_1(Y, y_0)$.
Then part (b) implies that for all $x_1 \in p^{-1}(y_0)$,

$$p_*(\pi_1(X, x_1)) = p_*(\pi_1(X, x_0))$$

So we can apply the theorem 8.26 to the cover
$p : (X, x_1) \to (Y, y_0)$ and the continuous morphism
$p : (X, x_0) \to (Y, y_0)$:

$$\begin{tikzcd}
 & (X, x_1) \arrow[d, "p"] \\
(X, x_0) \arrow[ur, dotted, "\exists f"] \arrow[r, "p"'] & (Y, y_0)
\end{tikzcd}$$

giving a continuous morphism $f : (X, x_0) \to (X, x_1)$.

It remains to see that f is a homeomorphism. Swapping the roles of (X, x_0) and (X, x_1) in the previous argument gives a continuous morphism $f^{-1}: (X, x_1) \longrightarrow (X, x_0)$.
Then id, $f^{-1} \circ f : X \longrightarrow X$ are continuous maps with

$$id(x_0) = x_0 = f^{-1} \circ f(x_0) \quad \text{and} \quad p \circ id = p = p \circ f^{-1} \circ f.$$

Thus the argument of part (a) (which does not use f to be a homeo) implies $id = f^{-1} \circ f$. Similarly $f \circ f^{-1} = id$.
$\Rightarrow$ f is a homeomorphism.

Assume now that $Aut(X \xrightarrow{p} Y)$ acts transitively on $p^{-1}(y_0)$.
Choosing some class $[\alpha] \in \pi_1(Y, y_0)$, we have to see that

$$[\alpha^{-1}] * p_*(\pi_1(X, x_0)) * [\alpha] \subseteq p_*(\pi_1(X, x_0)) \qquad (*)$$

Lift a representative $\alpha : I \longrightarrow Y$ of $[\alpha]$ to a path $\alpha' : I \longrightarrow X$ with $\alpha'(0) = x_0$. Then setting $x_1 = \alpha'(1)$, part (b) implies that $(*)$ is equivalent to

$$p_*(\pi_1(X, x_1)) \subseteq p_*(\pi_1(X, x_0)).$$

To prove this, choose a $f \in Aut(X \xrightarrow{p} Y)$ with $f(x_1) = x_0$ (possible by our assumption!) Then we have

$$\begin{tikzcd} & (X, x_0) \arrow[d, "p"] \\ (X, x_1) \arrow[ur, "f"] \arrow[r, "p"'] \arrow[ur, phantom, shift right=3, "\circlearrowright"] & (Y, y_0) \end{tikzcd}$$

i.e. f is a lift of $p : (X, x_1) \longrightarrow (Y, y_0)$. But by theorem 8.26 this implies directly

$$p_*(\pi_1(X, x_1)) \subseteq p_*(\pi_1(X, x_0))$$

as desired.

(d) Step 1: Definition of $\Phi : \pi_1(Y, y_0) \longrightarrow \mathrm{Aut}(X \xrightarrow{p} Y)$.

Let $[\gamma] \in \pi_1(Y, y_0)$ and represent it by some $\gamma : I \to Y$. Lift γ to a path $\gamma' : I \to X$ with $\gamma'(0) = x_0$. Then define $\Phi([\gamma])$ to be the unique $f \in \mathrm{Aut}(X \xrightarrow{p} Y)$ with $f(x_0) = x_1 := \gamma'(1)$.

Step 2: Φ is well-defined

We have to see that $\Phi([\gamma])$ does not depend on the choice of the representative γ. So let $\gamma_1, \gamma_2 : I \to Y$ be two such representatives. Then there exists a homotopy $H : I \times I \to Y$ from γ_1 to γ_2. By the homotopy lifting theorem 8.24 there is some $\tilde{H} : I \times I \longrightarrow X$ with $\tilde{H}|_{\{0\} \times I} = \gamma_1'$ and $p \circ \tilde{H} = H$. In particular $\tilde{H}|_{\{1\} \times I} : I \to X$ is a path satisfying

$p \circ \tilde{H}|_{\{1\} \times I} = H|_{\{1\} \times I} = \gamma_2$

i.e. by uniqueness of lifts $\tilde{H}|_{\{1\} \times I} = \gamma_2'$.

Next note that $\tilde{H}|_{I \times \{1\}}$ is a path from $\gamma_1'(1)$ to $\gamma_2'(1)$ lifting $H|_{I \times \{1\}} = c_{y_0}$ the constant path at y_0. By uniqueness of lifts, $\tilde{H}|_{I \times \{1\}}$ has to be the constant path at $\gamma_1'(1)$, i.e. $\gamma_2'(1) = \gamma_1'(1) = x_1$.

Step 3: Φ is a group morphism

Let $[\gamma], [\sigma] \in \pi_1(Y, y_0)$, $\gamma, \sigma : I \to Y$ representatives, $\gamma', \sigma' : I \to X$ lifts with $\gamma'(0) = x_0 = \sigma'(0)$ and $f_\gamma = \Phi([\gamma])$, $f_\sigma = \Phi([\sigma])$ the images in $\mathrm{Aut}(X \xrightarrow{p} Y)$.

Then $f_\gamma(\sigma') : I \longrightarrow X$ is a lift of σ starting at $f_\gamma(\sigma'(0)) = f_\gamma(x_0) = \gamma'(1)$. Hence $\gamma' * f_\gamma(\sigma') : I \to X$ is a well-defined path starting at x_0. As it lifts $\gamma * \sigma$, we have: $\gamma' * f_\gamma(\sigma') = (\gamma * \sigma)'$

$\Rightarrow \Phi([\gamma] \cdot [\sigma])(x_0) = (\gamma * \sigma)'(1) = (\gamma' * f_\gamma(\sigma'))(1) = f_\gamma(\sigma'(1)) =$

$= f_\gamma(f_\sigma(x_0)) = (f_\gamma \circ f_\sigma)(x_0) = (\Phi[\gamma] \cdot \Phi[\sigma])(x_0)$

Then by part (a):

$$\Phi([\gamma]\cdot[\sigma]) = \Phi[\gamma]\cdot\Phi[\sigma] \in \mathrm{Aut}(X \xrightarrow{p} Y)$$

Step 4: Φ is surjective

Let $f \in \mathrm{Aut}(X \xrightarrow{p} Y)$. Then choose a path $\gamma' : I \to X$ from x_0 to $f(x_0)$. Then $p(\gamma') \in \pi_1(Y, y_0)$. Moreover by construction

$$\Phi(p(\gamma'))(x_0) = \cancel{f(x_0)}\ \gamma'(1) = f(x_0)$$

$$\overset{(a)}{\Rightarrow} \Phi(p(\gamma')) = f$$

$\Rightarrow$ Φ surjective.

Step 5: $\ker \Phi \cong \pi_1(X, x_0)$

Consider $p_* : \pi_1(X, x_0) \longrightarrow (Y, y_0)$.

For any $[\gamma'] \in \pi_1(X, x_0)$ with representative $\gamma' : I \to X$, this γ' is a lift of $p(\gamma') : I \to Y$. Hence $\Phi(p_*[\gamma'])$ is the unique automorphism ~~with~~ mapping x_0 to $\gamma'(1) = x_0$.

$\Rightarrow \Phi(p_*[\gamma']) = id_X$

$\Rightarrow$ There is a well-defined map

$$p_* : \pi_1(X, x_0) \longrightarrow \ker \Phi.$$

Its inverse is given by

$$p_*^{-1} : \ker \Phi \longrightarrow \pi_1(X, x_0) \quad , \quad [\gamma] \longmapsto [\gamma']$$

where γ' is the lift of γ with $\gamma'(0) = x_0$. For $[\gamma] \in \ker \Phi$ we have $\gamma'(1) = x_0$, so p_*^{-1} is well-defined.

Moreover p_*^{-1} is really the inverse of p_*, which can be checked already on the level of paths (while ignoring all homotopies).

A 28)

(a) Let $x \in X$ and choose $x \in U \subseteq X$ s.th. $U \cap gU = \emptyset$ $\forall g \in G \setminus \{e\}$.

Then $q^{-1}(q(U)) = \bigcup_{g \in G} g.U$ is open, hence $q(U)$ is an open neighborhood of $q(x)$. Moreover for every $g \in G$

$$q : g.U \longrightarrow q(U) \quad \text{is a homeomorphism.}$$

It is clearly continuous and bijective. To check openness, let $V \subseteq g.U$ be open. Then $q^{-1}(q(V)) = \bigcup_{\tilde{g} \in G} \tilde{g}.V$ is open, hence $q(V)$ is open as desired.

$\Rightarrow$ $q : q^{-1}(q(U)) \longrightarrow q(U)$ is a homeo on connected components

$\Rightarrow$ q is a covering (because the $q(U)$ cover $q(X) = X/_G$).

(b) Consider

$$G \longrightarrow \mathrm{Aut}(X \xrightarrow{q} X/_G)$$
$$g \longmapsto g. : X \to X, \quad x \mapsto g.x$$

It is injective, because only $e \in G$ acts as the identity.
Moreover G acts transitively on $q^{-1}(q(x_0)) = G.x_0$. Hence for any $f \in \mathrm{Aut}(X \xrightarrow{q} X/_G)$, there is a $g \in G$ with $f(x_0) = g.x_0$. Hence by A27(a): $f = g. \in \mathrm{Aut}(X \xrightarrow{q} X/_G)$.

$\Rightarrow$ $G \cong \mathrm{Aut}(X \xrightarrow{q} X/_G)$ is an isomorphism

The arguments above show that A27(c) are satisfied, hence by A27(d):

$$\pi_1(X/_G, q(x_0)) \xrightarrow{\Phi} \mathrm{Aut}(X \xrightarrow{q} X/_G) \cong G$$

is surjective with kernel isomorphic to $\pi_1(X, x_0)$.

A29)

(a) Consider $p : \mathbb{R}^2 \longrightarrow K$

$$(x,y) \longmapsto \begin{cases} (x \bmod 1, \ y \bmod 1) & \text{if } x \in [2n, 2n+1), n \in \mathbb{Z} \\ (x \bmod 1, \ -y \bmod 1) & \text{if } x \in [2n+1, 2n+2), n \in \mathbb{Z}. \end{cases}$$

One can check by direct computation that p is a covering.

To compute $\mathrm{Aut}(\mathbb{R}^2 \xrightarrow{p} K)$, consider the following two deck transforms:

$\sigma : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x,y) \longmapsto (x, y+1)$

$\tau : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x,y) \longmapsto (x+1, -y)$

Indeed by construction $p \circ \sigma = p = p \circ \tau : \mathbb{R}^2 \to K$.

Claim: $\mathrm{Aut}(\mathbb{R}^2 \xrightarrow{p} K) = \{ \sigma^a \cdot \tau^b \ , a, b \in \mathbb{Z} \}$

Proof: Taking $x_0 = (0,0)$, it suffices to show that $\{\sigma^a \tau^b\}$ acts transitively on $p^{-1}(x_0) = \mathbb{Z}^2 \subseteq \mathbb{R}^2$. But

$\sigma^a \tau^b (0,0) = \sigma^a (b, 0) = (b, a)$

So the claim follows.

So the assumptions of A27 (c) are satisfied and we may apply A27(d). With $\pi_1(\mathbb{R}^2, (0,0)) = 1$, we obtain:

$$\pi_1(K, (0,0)) \cong \mathrm{Aut}(\mathbb{R}^2 \xrightarrow{p} K) \cong \{ \sigma^a \tau^b \mid a, b \in \mathbb{Z} \}.$$

In this last group, the multiplication is given by

$$(\sigma^a \tau^b) \cdot (\sigma^c \tau^d) = \sigma^{a + (-1)^b c} \cdot \tau^{b+d}.$$