Solutions to Sheet 11

A30 a) To distinguish the groups $\mathbb{Z}$, write $\mathbb{Z} = \{a^n, n \in \mathbb{Z}\}$ for the target group of f_1 and $\mathbb{Z} = \{b^n, n \in \mathbb{Z}\}$ for the target group of f_2.

Moreover denote by $\langle S \rangle_N$ the normal closure of a subset $S \subseteq G$.

Then by definition:

$$\mathbb{Z} *_{\mathbb{Z}\times\mathbb{Z}} \mathbb{Z} = \mathbb{Z} * \mathbb{Z} / \langle f_1(x,y) \cdot f_2(x,y)^{-1}, (x,y) \in \mathbb{Z}\times\mathbb{Z} \rangle_N$$

$$= \mathbb{Z} * \mathbb{Z} / \langle a^{2x+3y} \cdot b^{-3x-2y}, (x,y) \in \mathbb{Z}\times\mathbb{Z} \rangle_N$$

Plugging in $x = -1$, $y = 1$, we see that $a \cdot b^{-1}$ lies in the subgroup we are dividing out. So

$$= \left(\mathbb{Z} * \mathbb{Z} / \langle a \cdot b^{-1} \rangle_N \right) / \langle a^{2x+3y} \cdot b^{-3x-2y} \rangle_N$$

Now

$$\mathbb{Z} * \mathbb{Z} / \langle a \cdot b^{-1} \rangle_N \longrightarrow \mathbb{Z} = \{c^n, n \in \mathbb{Z}\}$$

$$a \longmapsto c$$

$$b \longmapsto c^{-1}$$

is a well-defined morphism. By a direct computation one sees that it is bijective. Moreover $a^{2x+3y} \cdot b^{-3x-2y}$ gets mapped to c^{5x+5y}. Hence we get:

$$\mathbb{Z} *_{\mathbb{Z}\times\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z} / \langle c^{5x+5y} \rangle_N = \mathbb{Z}/5\mathbb{Z}.$$

b) We have $G_1 * \{e\} \cong G_1$, via $g \mapsto g \;\forall g \in G_1$, $e \mapsto e_{G_1}$ map the unit element of $\{e\}$ to the unit element in G_1. Then

$$G_1 *_{G_0} \{e\} = G_1 * \{e\} / \langle f(g) \cdot e^{-1}, g \in G_0 \rangle_N \cong$$

$$\cong G_1 / \langle f(g), g \in G_0 \rangle_N =$$

$$= G_1 / \langle f(G_0) \rangle_N = G_1 / H.$$

A 31) Consider the cover of K given by

$$U = K \setminus \{(\tfrac{1}{2}, \tfrac{1}{2})\} \quad , \quad V = (0,1) \times (0,1)$$

Then we have homotopy equivalences:

$V \simeq pt.$

$U \cap V = \big((0,1)\times(0,1)\big) \setminus \{(\tfrac{1}{2},\tfrac{1}{2})\} \simeq S^1$

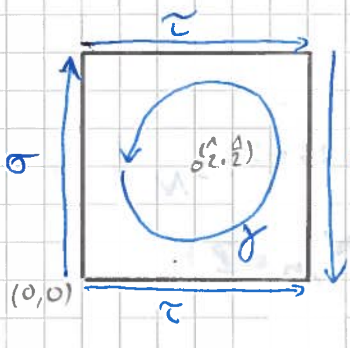
$U \simeq (I \times \{0,1\} \cup \{0,1\} \times I)/\sim \;\cong\; I \times \{0\} \cup \{0\} \times I /_{(0,0)\sim(1,0)\sim(0,1)}$

$\cong S^1 \vee S^1$

$\Rightarrow \pi_1(V) = \{e\}$

$\pi_1(U \cap V) \cong \mathbb{Z}$ with generator γ

$\pi_1(U) = \mathbb{Z} * \mathbb{Z}$ with generators σ, τ



Now $\pi_1(U \cap V) \longrightarrow \pi_1(V)$, $\gamma \longmapsto 1$

because this is the only existing morphism to the trivial group

and $\pi_1(U \cap V) \longrightarrow \pi_1(U)$, $\gamma \longmapsto \tau\sigma^{-1}\tau^{-1}\sigma^{-1}$

by deforming γ to the boundary of the square

So by Seifert - van Kampen

$$\begin{tikzcd} \pi_1(U \cap V) \cong \mathbb{Z} \arrow[r] \arrow[d, "\gamma \mapsto \tau\sigma^{-1}\tau^{-1}\sigma^{-1}"'] & \pi_1(V) = \{e\} \arrow[d] \\ \pi_1(U) \arrow[r] & \pi_1(K) \end{tikzcd}$$

$\Rightarrow \pi_1(K) \cong (\mathbb{Z} * \mathbb{Z}) \underset{\mathbb{Z}}{*} \{e\} \cong \mathbb{Z} * \mathbb{Z} / \langle \tau\sigma^{-1}\tau^{-1}\sigma^{-1} \rangle_N$

(by exercise A30b)).

Remark: We compare it to the description of $\pi_1(K)$ obtained in exercise A 29 a) via the isomorphism:

$$\mathbb{Z} * \mathbb{Z} / \langle \tau\sigma^{-1}\tau^{-1}\sigma^{-1} \rangle_N \longrightarrow \{\sigma^a \tau^b \mid a, b \in \mathbb{Z}\}$$
$$\sigma \longmapsto \sigma$$
$$\tau \longmapsto \tau .$$

A 32)

a) Solution 1: Universal covering

The group $\mathbb{Z}/2\mathbb{Z}$ acts on S^n via $x \mapsto -x$. Then by definition $\mathbb{R}P^n \cong S^n/(\mathbb{Z}/2\mathbb{Z})$. By taking small ε-balls around points, the assumptions in ex. A 28) are satisfied.

$\Rightarrow$ $\pi_1(\mathbb{R}P^n) \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is surjective with kernel $\pi_1(S^n) = \{e\}$

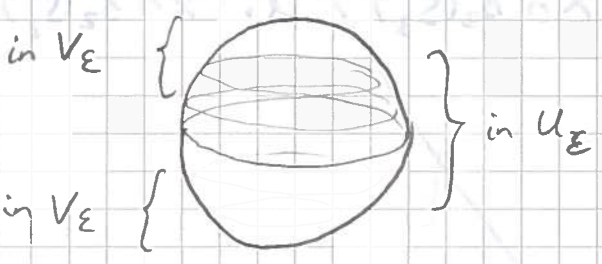
$\Rightarrow$ $\pi_1(\mathbb{R}P^n) \xrightarrow{\sim} \mathbb{Z}/2\mathbb{Z}$ is an isomorphism.

Solution 2: Seifert-van Kampen.

Let $q: S^n \longrightarrow \mathbb{R}P^n$ be the quotient morphism and define $U_\varepsilon = \{(x_0, \dots, x_n) \in S^n \mid |x_n| < 2\varepsilon\}$

$V_\varepsilon = \{(x_0, \dots, x_n) \in S^n \mid |x_n| > \varepsilon\}$

for some very small ε.



n-dim. open ball

Then • $q(V_\varepsilon) \cong \{(x_0, \dots, x_n) \in S^n \mid x_n > \varepsilon\} \cong B_1(0)$

• $U_\varepsilon \longrightarrow \{(x_0, \dots, x_n) \in S^n \mid x_n = 0\} \cong S^{n-1}$

$(x_0, \dots, x_n) \longmapsto \frac{1}{\|(x_0, \dots, x_{n-1})\|}(x_0, \dots, x_{n-1}, 0)$

is a homotopy equivalence compatible with $x \mapsto -x$ on both sides. Hence $q(U_\varepsilon) \simeq S^{n-1}/(x \sim -x) = \mathbb{R}P^{n-1}$

• $q(U_\varepsilon \cap V_\varepsilon) \cong \{(x_0, \dots, x_n) \in S^n \mid \varepsilon < x_n < 2\varepsilon\} \simeq S^{n-1}$

homotopy equivalence.

Case 1: $n=2$.

Then $\pi_1(q(V_\varepsilon)) \cong \pi_1(B_1(0)) = \{e\}$

$\pi_1(q(U_\varepsilon)) \cong \pi_1(\mathbb{R}P^1) = \pi_1(S^1) \cong \mathbb{Z}$

$\pi_1(q(U_\varepsilon \cap V_\varepsilon)) \cong \pi_1(S^1) \cong \mathbb{Z}$.

Now a generator of $\pi_1(q(U_\varepsilon \cap V_\varepsilon))$ is given by going around the equator once. But this corresponds to going round ~~the~~ twice the S^1 under the isomorphism $\pi_1(q(U_\varepsilon)) \cong \pi_1(S^1)$.

$$\Rightarrow \quad \begin{tikzcd} \pi_1(q(U_\varepsilon \cap V_\varepsilon)) \cong \mathbb{Z} \arrow[r] \arrow[d, "\cdot 2"'] & \pi_1(q(V_\varepsilon)) = \{e\} \arrow[d] \\ \pi_1(q(U_\varepsilon)) \cong \mathbb{Z} \arrow[r] & \pi_1(\mathbb{R}P^2) \end{tikzcd}$$

$\Rightarrow \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$.

Case 2: $n \geq 3$.

Then $\pi_1(q(V_\varepsilon)) \cong \pi_1(B_1(0)) = \{e\}$

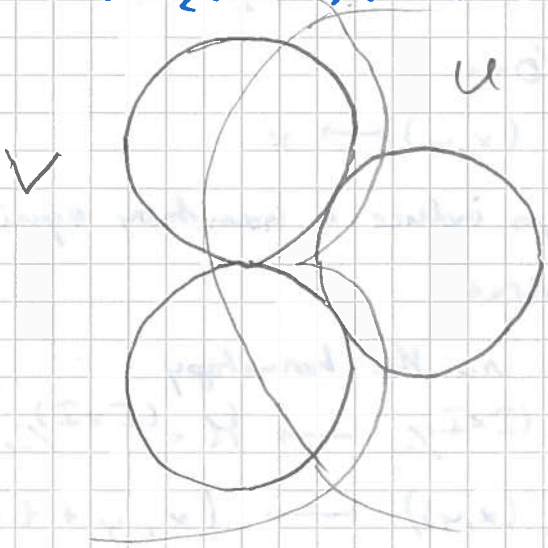
$\pi_1(q(U_\varepsilon)) \cong \pi_1(\mathbb{R}P^{n-1}) \cong \mathbb{Z}/2\mathbb{Z}$ by induction

$\pi_1(q(U_\varepsilon \cap V_\varepsilon)) \cong \pi_1(S^{n-1}) \cong \{e\}$

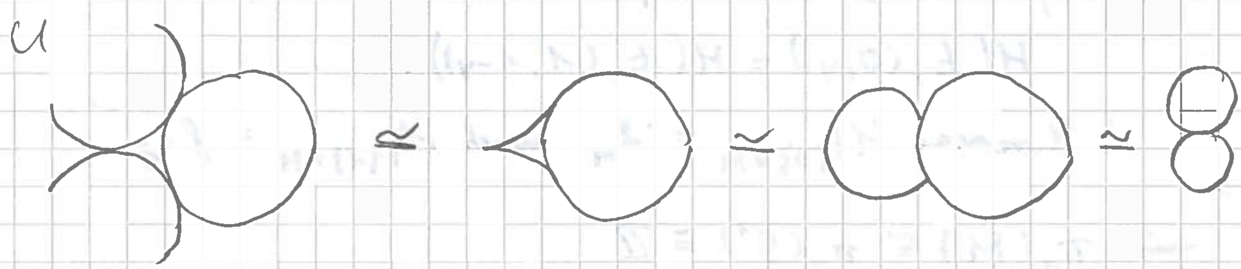
$\Rightarrow \pi_1(\mathbb{R}P^n) \cong \pi_1(\mathbb{R}P^{n-1}) *_{\{e\}} \{e\} = \pi_1(\mathbb{R}P^{n-1}) \cong \mathbb{Z}/2\mathbb{Z}$.

ii) We apply Seifert - van Kampen to

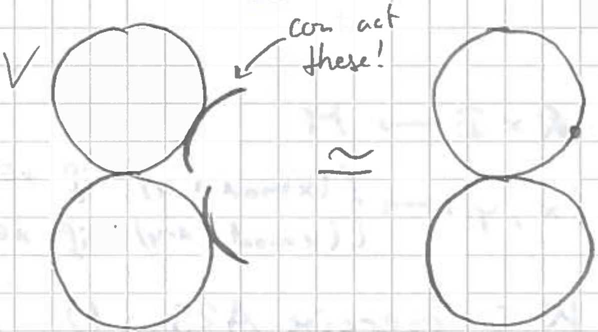
$U = X \cap B_2(1)$, $V = (X \cap B_1(S_3)) \cup (X \cap B_1(S_3^2))$



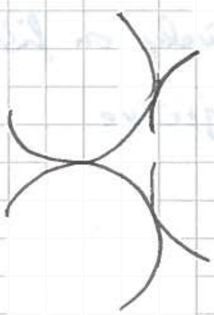
Now $U \simeq S^1 \vee S^1$:



and $V \simeq S^1 \vee S^1$:



and $U \cap V \simeq pt$



is a tree, hence contractible.

$\Rightarrow \pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \cong \pi_1(S^1 \vee S^1) *_{\pi_1(pt)} \pi_1(S^1 \vee S^1)$

$\cong (\mathbb{Z} * \mathbb{Z}) * (\mathbb{Z} * \mathbb{Z})$

is the free group in 4 generators.

iii) Solution 1: Homotopies

Consider $f: S^1 \longrightarrow M$, $x \longmapsto (x, \frac{1}{2})$, where we use the description $S^1 = I/_{0\sim 1}$.

and $g: M \longrightarrow S^1$, $(x,y) \longmapsto x$.

We claim that these maps induce a homotopy equivalence:

$g \circ f = id_{S^1}$ is obvious

$f \circ g \simeq id_M$ is true via the homotopy

$$H: I \times M = I \times {}^{(I \times I)}/_{\sim} \longrightarrow M = {}^{(I \times I)}/_{\sim}$$

$$(t, (x,y)) \longmapsto (x, y + t(\tfrac{1}{2} - y))$$

One immediately checks, that this map respects the equivalence relation, i.e. that

$$H(t,(0,y)) = H(t,(1,1-y)).$$

Moreover $H|_{\{0\}\times M} = id_M$ and $H|_{\{1\}\times M} = f \circ g$.

$\longrightarrow \pi_1(M) \cong \pi_1(S^1) \cong \mathbb{Z}$

Solution 2: Universal covering

Consider ~~$p: \mathbb{R} \longrightarrow$~~ $p: \mathbb{R} \times I \longrightarrow M$

$$(x,y) \longmapsto \begin{cases} (x \bmod 1, y) & \text{if } x \in [2n, 2n+1) \\ (x \bmod 1, 1-y) & \text{if } x \in [2n+1, 2n+2) \end{cases}$$

(compare with the cover of K in exercise A25a) !)

Then one can describe $Aut(\mathbb{R} \times I \xrightarrow{p} M) \cong \mathbb{Z}$ explicitly and check that it acts transitively on fibers.

$\Longrightarrow$ $\pi_1(M) \longrightarrow Aut(\mathbb{R} \times I \xrightarrow{p} M) \cong \mathbb{Z}$ surjective with kernel $\pi_1(\mathbb{R} \times I) = \{e\}$.

$\Longrightarrow$ $\pi_1(M) \cong Aut(\mathbb{R} \times I \xrightarrow{p} M) \cong \mathbb{Z}$.