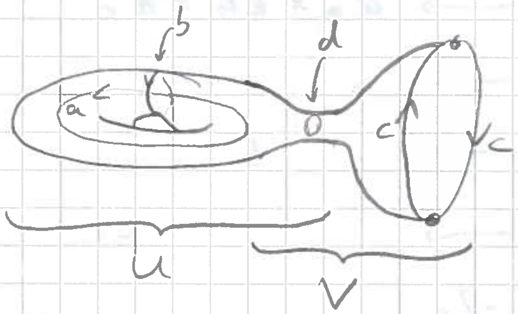
Solutions for Sheet 12

A 33)

a)



Then U = torus without a small disk

V = $\mathbb{R}P^2$ without a small disc.

$U \cap V$ = cylinder connecting the torus and $\mathbb{R}P^2$.

$$\Rightarrow \pi_1(U) \cong \pi_1\left(I\times\{0,1\} \cup \{0,1\}\times I \,/\!\sim \right) \cong \pi_1(S^1 \vee S^1) = \langle a, b \rangle$$

$$\pi_1(V) \cong \pi_1\left(\overline{B}_1(0) \setminus B_\varepsilon(0) \,/\!\sim \right) \cong \pi_1\left(S^1/_{x\sim -x} \right) \cong \pi_1(S^1) \cong \langle c \rangle$$

this identifies x and $-x$ on $\partial \overline{B}_1(0)$.

$$\pi_1(U \cap V) \cong \pi_1(S^1) \cong \cancel{\mathbb{Z}} \langle d \rangle$$

Now
$$\pi_1(U\cap V) \longrightarrow \pi_1(U)$$
$$d \longmapsto a b a^{-1} b^{-1}$$
$$\pi_1(U\cap V) \longrightarrow \pi_1(V)$$
$$d \longmapsto c^2$$

So by Seifert-van Kampen:

$$\pi_1(X) = \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) = \langle a,b,c \rangle / \langle aba^{-1}b^{-1} \cdot c^{-2} \rangle .$$

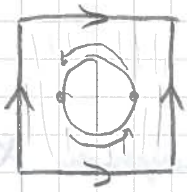
Remark:

Alternatively view $X = \cancel{I\times I/\!\sim}\; I\times I \setminus B_{\frac{1}{4}}(\tfrac{1}{2},\tfrac{1}{2}) /\!\sim$

where $(x,0) \sim (x,1) \quad \forall x \in I$

$(0,y) \sim (1,y) \quad \forall y \in I$

$(x,y) \sim (1-x, 1-y) \quad \forall (x,y) \in \partial B_{\frac{1}{4}}(\tfrac{1}{2},\tfrac{1}{2})$



and cover it with

$U = X \setminus \overline{B}_{\frac{1}{3}}(\tfrac{1}{2},\tfrac{1}{2})$.

$V = B_{2}(\tfrac{1}{2},\tfrac{1}{2}) \setminus B_{\frac{1}{4}}(\tfrac{1}{2},\tfrac{1}{2})$

b) We have the abelianization morphism

$$\langle a,b,c\rangle \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} \longrightarrow \mathbb{Z}a \times \mathbb{Z}b \times \mathbb{Z}c$$

$$a \longmapsto a$$

$$b \longmapsto b$$

$$c \longmapsto c$$

mapping $aba^{-1}b^{-1}c^{-2}$ to $-2c$.

So $\left(\langle a,b,c\rangle / \langle aba^{-1}b^{-1}c^{-2}\rangle\right)^{ab} \cong \langle a,b,c\rangle^{ab} / -2c =$

$$= \mathbb{Z}a \times \mathbb{Z}b \times \mathbb{Z}c / \langle -2c\rangle \cong \mathbb{Z} \times \mathbb{Z} \times (\mathbb{Z}/2).$$

c) It suffices to show $\pi_1(X)^{ab} \not\cong \pi_1(S_g)^{ab}$ and $\pi_1(X)^{ab} \not\cong \pi_1(N_h)^{ab}$ for $h \neq 3$.

By the lecture: $\pi_1(S_g)^{ab} \cong \mathbb{Z}^{2g}$

$\pi_1(N_h)^{ab} \cong \mathbb{Z}^h / (2,2,\dots,2)$.

$\rightsquigarrow$ $\pi_1(X)^{ab} \not\cong \pi_1(S_g)^{ab}$ because $\pi_1(S_g)^{ab}$ has no element of order 2, while $\pi_1(X)^{ab}$ has.

$\rightsquigarrow$ $\pi_1(N_h)^{ab} \cong \mathbb{Z}^h/(2,2,\dots,2) \cong \mathbb{Z}^{h-1} \times (\mathbb{Z}/2)$

$$(x_1,\dots,x_{h-1},x_h) \longmapsto (x_1-x_h,\, x_2-x_h,\dots,x_{h-1}-x_h,\, x_h \bmod 2)$$

So $\pi_1(X)^{ab} \not\cong \pi_1(N_h)^{ab}$ for $h \neq 3$ because they have a different number of $\mathbb{Z}$-factors.

Remark:

There is an isomorphism

$$\pi_1(X) = \langle a,b,c\rangle / \langle aba^{-1}b^{-1}c^{-2}\rangle \xrightarrow{\sim} \pi_1(N_3) = \langle x,y,z\rangle / \langle x^2y^2z^2\rangle$$

$$a \longmapsto xy$$

$$b \longmapsto z^{-1}y^{-2}x^{-1}$$

$$c \longmapsto xyz$$

$$cba \longleftarrow x$$

$$a^{-1}b^{-1}c^{-1}a \longleftarrow y$$

$$a^{-1}c \longleftarrow z$$

In fact one can even construct a homeomorphism $X \cong N_3$.

34) a) Choose a trivializing open cover $\{U_i\}$ of B. We claim that $\{g^{-1}(U_i)\}$ is a trivializing open cover for B'.

So fix a trivialization of X over U_i:

$$\begin{tikzcd} f^{-1}(U_i) \arrow[rr, "\sim"] \arrow[dr] & & U_i \times F \arrow[dl] \\ & U_i & \end{tikzcd}$$

Base-changing this diagram via $\ldots \times_B B'$ gives

$$\begin{tikzcd} f^{-1}(U_i) \times_B B' \arrow[rr, "\sim"] \arrow[dr] & & (U_i \times F) \times_B B' \arrow[dl] \\ & U_i \times_B B' & \end{tikzcd}$$

Now note that $U_i \hookrightarrow B$ is an embedding. Hence

$$U_i \times_B B' \cong g^{-1}(U_i)$$

Similarly

$$f^{-1}(U_i) \cong U_i \times_B X$$

$$\Rightarrow f^{-1}(U_i) \times_B B' \cong (X \times_B U_i) \times_B B' \cong X \times_B (U_i \times_B B')$$
$$\cong X \times_B g^{-1}(U_i) \cong (X \times_B B') \times_{B'} g^{-1}(U_i)$$
$$= X' \times_{B'} g^{-1}(U_i) \cong f'^{-1}(g^{-1}(U_i)).$$

and finally

$$(U_i \times F) \times_B B' \cong (U_i \times_B B') \times F \cong g^{-1}(U_i) \times F$$

Thus the diagram above is equivalent to

$$\begin{tikzcd} f'^{-1}(g^{-1}(U_i)) \arrow[rr, "\sim"] \arrow[dr] & & g^{-1}(U_i) \times F \arrow[dl] \\ & g^{-1}(U_i) & \end{tikzcd}$$

and the claim follows.

b) Consider some point $x \in B$. Choose an elementary neighborhood $U_x \subseteq B$, a point in the fiber $y \in g^{-1}(B)$ and a trivializing neighborhood $V_y \subseteq B'$ for f'. Set $W_y = V_y \cap U_y$, where U_y is the connected component of $g^{-1}(U_x)$ containing y.

$\Rightarrow g|_{W_y} : W_y \longrightarrow U_x$ is a homeo onto an open subset.

Now by construction of X', $g|_{W_y}$ induces a~~n iso~~ homeomorphism

$$\begin{array}{ccccc} X' \supseteq & f'^{-1}(W_y) & \xrightarrow{\sim} & f^{-1}(g(W_y)) & \subseteq X \\ & \downarrow & & \downarrow & \\ & W_y & \xrightarrow{\cong} & g(W_y) & \end{array}$$

So any trivialization $f'^{-1}(W_y) \cong W_y \times F$ induces a trivialization $f^{-1}(g(W_y)) \cong g(W_y) \times F$.

$\Rightarrow f : X \longrightarrow B$ is a fiber bundle with fiber F.

A35) As suggested by the hint, define for $W \subseteq \mathbb{C}^n$ of dim. k:

$$\mathcal{U}_W = \{ V \in Gr_k(\mathbb{C}^n) \mid V \cap W^\perp = 0 \} \subseteq Gr_k(\mathbb{C}^n).$$

Claim 1: $\mathcal{U}_W$ is open in $Gr_k(\mathbb{C}^n)$.

Choose a basis $w_{k+1}, \dots, w_n$ of $W^\perp$. Then

$$\begin{aligned} \tilde{\mathcal{U}}_W &= \{ (v_1, \dots, v_k) \in (\mathbb{C}^n)^k \mid \langle v_1, \dots, v_k \rangle \text{ are lin. indep. and } \cancel{\det} \langle v_1, \dots, v_k \rangle \cap W^\perp = 0 \} \\ &= \{ (v_1, \dots, v_k) \in (\mathbb{C}^n)^k \mid \det (v_1, \dots, v_k, w_{k+1}, \dots, w_n) \neq 0 \} \end{aligned}$$

is open in $(\mathbb{C}^n)^k$.

Recall the Stiefel manifold

$$V_k = \{ (v_1, \dots, v_k) \in (\mathbb{C}^n)^k \mid v_i\text{'s are orthonormal vectors} \}$$

Then we have the quotient map

$$V_k \xrightarrow{q} Gr_k(\mathbb{C}^n) \quad , \quad (v_1, \dots, v_k) \longmapsto \langle v_1, \dots, v_k \rangle$$

So

$$\begin{aligned} q^{-1}(\mathcal{U}_W) &= \{ (v_1, \dots, v_k) \in V_k \mid \langle v_1, \dots, v_k \rangle \cap W^\perp = 0 \} \\ &= V_k \cap \tilde{\mathcal{U}}_W \end{aligned}$$

which is open in V_k

$\Rightarrow \mathcal{U}_W$ is open in $Gr_k(\mathbb{C}^n)$.

Claim 2: X trivializes over U_W

Let $pr_W : \mathbb{C}^n \to W$ be the orthogonal projection onto W.

Then define:

$$\begin{aligned} pr^{-1}(U_W) &\longrightarrow U_W \times W \cong U_W \times \mathbb{C}^k \\ (V, v) &\longmapsto (V, pr_W v) \end{aligned}$$

It is continuous. On each fiber over a point in U_W it is an injective linear map, hence an isomorphism.

Moreover using the orthogonal projection $pr_V : \mathbb{C}^n \to V$ onto V we can give its inverse by

$$\begin{aligned} U_W \times W &\longrightarrow pr^{-1}(U_W) \\ (V, w) &\longmapsto \left(V, \frac{\langle w, w\rangle}{\langle pr_V w, w\rangle} \cdot pr_V w\right) \end{aligned}$$

So $pr^{-1}(U_W) \cong U_W \times W$ is a homeomorphism and X is a fiber bundle over U_W with typical fiber $\mathbb{C}^k$.