Solutions for Sheet 13

A 36)

a) In addition to the atlas $\{U_i, g_i : f^{-1}(U_i) \xrightarrow{\sim} U_i \times F\}$ choose a global trivialization $h: X \longrightarrow B \times F$.

Then for $U_{ij} = U_i \cap U_j$:

$$g_j \circ g_i^{-1} = g_j \circ h^{-1}|_{f^{-1}(U_{ij})} \circ h|_{f^{-1}(U_{ij})} \circ g_i^{-1} =$$

$$= \left(h|_{f^{-1}(U_{ij})} \circ g_j^{-1}\right)^{-1} \circ \left(h|_{f^{-1}(U_{ij})} \circ g_i^{-1}\right)^{-1}$$

Hence:

$$t_{ij} = (g_j \circ g_i^{-1})^{\#} = \left(h|_{f^{-1}(U_j)} \circ g_j^{-1}\right)|_{U_{ij}}^{-1\#} \cdot \left(h|_{f^{-1}(U_i)} \circ g_i^{-1}\right)^{\#}|_{U_{ij}} :$$

$$U_{ij} \longrightarrow \mathrm{Aut}(F)$$

So setting $\psi_i = \left(h|_{f^{-1}(U_i)} \circ g_i^{-1}\right)^{\#} : U_i \to \mathrm{Aut}(F)$ gives the desired functions with $t_{ij} = \psi_j^{-1} \cdot \psi_i$.

b) Assume there are $\psi_i : U_i \to \mathrm{Aut}(F)$ with $t_{ij} = \psi_j^{-1} \cdot \psi_i$.

Then reverse the #-operation to get

$$\tilde{\psi}_i : U_i \times F \longrightarrow U_i \times F$$

Set $h_i = \tilde{\psi}_i \circ g_i : f^{-1}(U_i) \longrightarrow U_i \times F$.

Now observe that

$$(g_j \circ g_i^{-1})^{\#} = t_{ij} = \psi_j^{-1} \cdot \psi_i : U_{ij} \longrightarrow \mathrm{Aut}(F)$$

hence after reversing the #-operation:

$$g_j \circ g_i^{-1} = \tilde{\psi}_j^{-1} \circ \tilde{\psi}_i : U_{ij} \times F \longrightarrow U_{ij} \times F$$

$$\Rightarrow h_i = \tilde{\psi}_i \circ g_i = \tilde{\psi}_j \circ g_j = h_j : f^{-1}(U_{ij}) \longrightarrow U_{ij} \times F$$

This shows that the following morphism is well-defined

$$h: X \longrightarrow B \times F, \quad h(x) = h_i(x) \text{ for some } i \text{ such that } f(x) \in U_i$$

It is a homeomorphism, because this is true on the preimages of any U_i.

$\Rightarrow X$ is the trivial fiber bundle.

A37)

a) $\pi_1(S^3) = \{1\}$ and $\pi_1(S^2 \times S^1) \cong \pi_1(S^2) \times \pi_1(S^1) \cong \{1\} \times \mathbb{Z} \cong \mathbb{Z}$.
So $S^3 \not\cong S^2 \times S^1$ are not homeomorphic. Thus they cannot be isomorphic as bundles over S^2.

Alternative solution using A36):
If $h: S^3 \to S^2$ would be trivial, there would be

$$\psi_1 : \mathbb{C} \longrightarrow \mathrm{Aut}_{gp}(S^1) = S^1$$
$$\psi_2 : (\mathbb{C} \setminus \{0\}) \cup \{\infty\} \longrightarrow S^1$$

s.th. $\psi_2^{-1} \cdot \psi_1 = t_{12}$, where t_{ij} is the transition function for the trivializations given in the lecture, i.e.

$$\begin{aligned} t_{12} : \mathbb{C} \setminus \{0\} &\longrightarrow S^1 \\ \lambda &\longmapsto \left\{\frac{\lambda}{|\lambda|}\right\}. \end{aligned}$$

Now observe that $\mathbb{C}$ and $(\mathbb{C} \setminus \{0\}) \cup \{\infty\}$ are contractible, hence ψ_1, ψ_2 are homotopic to a constant map.
$\Rightarrow t_{12} = \psi_2^{-1} \cdot \psi_1$ is homotopic to a constant map.
On the other hand

$$\begin{aligned} t_{12}|_{S^1} : S^1 \hookrightarrow \mathbb{C} \setminus \{0\} &\longrightarrow S^1 \\ \lambda &\longmapsto \lambda \end{aligned}$$

is a generator of $\pi_1(S^1)$, hence certainly not homotopic to the identity. Contradiction.

b) As $h(z, z') = \frac{z}{z'} = \frac{\zeta_n z}{\zeta_n z'} = h(\zeta_n z, \zeta_n z')$, h factors over L_n inducing the morphism

$$h_n : L_n \longrightarrow S^2.$$

We show next, that the trivializations of h descend to trivializations of h_n, i.e. that $L_n \to S^2$ is a fibre bundle.

Recall the trivialization of h on $h^{-1}(\mathbb{C})$:

$$g_1 : h^{-1}(\mathbb{C}) \longrightarrow \mathbb{C} \times S^1$$
$$(z, z') \longmapsto \left(\frac{z}{z'}, \frac{z'}{|z'|}\right)$$

Then $g_1(\zeta_n z, \zeta_n z') = \left(\dfrac{\zeta_n z}{\zeta_n z'}, \dfrac{\zeta_n z'}{|\zeta_n z'|}\right) = \left(\dfrac{z}{z'}, \zeta_n \dfrac{z'}{|z'|}\right)$

Hence the composition

$$h^{-1}(\mathbb{C}) \xrightarrow{g_1} \mathbb{C} \times S^1 \xrightarrow{id \times (\,)^n} \mathbb{C} \times S^1/\{1, \zeta_n, \zeta_n^2, \dots\} \cong \mathbb{C} \times S^1$$
$$(x, t) \longmapsto (x, [t]) \longmapsto (x, t^n)$$

is constant on points in the same equivalence class of $h^{-1}(\mathbb{C})$. So we get an induced morphism on $h_n^{-1}(\mathbb{C}) \subseteq L_n$:

$$\tilde{g}_1 : h_n^{-1}(\mathbb{C}) \longrightarrow \mathbb{C} \times S^1.$$

By construction $\tilde{g}_1$ is continuous and bijective.
The same arguments give as well that $\tilde{g}_1^{-1}$ is continuous, hence $\tilde{g}_1$ is a homeomorphism.

In the same way one proves that h_n trivializes over $(\mathbb{C}\setminus\{0\}) \cup \{\infty\}$.

We still have to show that the transition morphisms map into $S^1 \subseteq Aut(S^1)$. So consider

$$\begin{array}{ccccl}
t_{12} = g_2 g_1^{-1} : (\mathbb{C}\setminus\{0\}) \times S^1 & \longrightarrow & (\mathbb{C}\setminus\{0\}) \times S^1 & & (\lambda, x) \mapsto (\lambda, \frac{\lambda}{|\lambda|} x) \\
\downarrow & & \downarrow & & \\
(\mathbb{C}\setminus\{0\}) \times S^1/\langle \zeta_n \rangle & \longrightarrow & (\mathbb{C}\setminus\{0\}) \times S^1/\langle \zeta_n \rangle & & (\lambda, x) \mapsto (\lambda, \frac{\lambda}{|\lambda|} x) \\
\simeq \downarrow & & \simeq \downarrow & & \\
(\mathbb{C}\setminus\{0\}) \times S^1 & \longrightarrow & (\mathbb{C}\setminus\{0\}) \times S^1 & & (\lambda, x) \mapsto (\lambda, \frac{\lambda^n}{|\lambda|^n} x)
\end{array}$$

where the last row is the transition morphism for the trivializations of $h_n : L_n \longrightarrow S^2$.
As multiplication with $\left(\frac{\lambda}{|\lambda|}\right)^n$ is in $S^1 \subseteq Aut(S^1)$, this shows that $L_n \longrightarrow S^2$ is indeed a S^1-bundle.

c) Let us compute $\pi_1(L_n)$:

Let $G = \mathbb{Z}/n\mathbb{Z}$ act on S^3 via

$$k.(z_1,z_2) = (\zeta_n^k z_1, \zeta_n^k z_2) \quad \text{for } k \in \mathbb{Z}/n\mathbb{Z},\ (z_1,z_2) \in S^3.$$

Then $L_n = S^3/G$.

Moreover for ε sufficiently small, the $B_\varepsilon(\zeta_n^k z_1, \zeta_n^k z_2)$ are disjoint for varying $k \in \mathbb{Z}/n\mathbb{Z}$ (as orbits are finite sets in $\mathbb{C}^2$).

So we may apply A2 8 b) and use $\pi_1(S^3) = \{1\}$ to get:

$$\pi_1(L_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

So all fibrations $L_n \longrightarrow S^2$ have different fundamental group, hence cannot be homeomorphic.