Solutions to sheet 2

A4) (a) It remains to see that $f^{-1}: Y \to X$ is continuous. So pick any $U \subseteq X$ open. Then by bijectivity of f:

$$(f^{-1})^{-1}(U) = f(U)$$

which is open, because f is open.

(b) Consider $f: \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^m \times \mathbb{R}^n$

$$(x_1, \dots, x_{m+n}) \longmapsto ((x_1, \dots, x_m), (x_{m+1}, \dots, x_{m+n}))$$

- f is bijective: obvious
- f is continuous: It suffices to check this on a subbasis. But in $\mathbb{R}^{m+n}$ all subsets of the form $U \times \mathbb{R}^n$ ($U \subseteq \mathbb{R}^m$ open) and $\mathbb{R}^m \times V$ ($V \subseteq \mathbb{R}^n$ open) are open.
- f is open: Again it suffices to check this on a subbasis. So taking any open ball $B_\delta(\underline{x}) \subseteq \mathbb{R}^{m+n}$, we have to construct for any point $\underline{y} \in B_\delta(\underline{x})$ an open neighbourhood of $f(\underline{y})$ contained in $f(B_\delta(\underline{x}))$.
  For this pick $\varepsilon > 0$ s.th. $\underline{y} \in B_\varepsilon(\underline{y}) \subseteq B_\delta(x)$ in $\mathbb{R}^{m+n}$.
  Then a direct calculation shows
  $$(B_{\varepsilon/2}(\underline{y}_1) \times \mathbb{R}^n) \cap (\mathbb{R}^m \times B_{\varepsilon/2}(\underline{y}_2)) \subseteq f(B_\varepsilon(\underline{y}))$$
  where $f(\underline{y}) = (\underline{y}_1, \underline{y}_2) \in \mathbb{R}^m \times \mathbb{R}^n$, giving the desired open neighbourhood of $f(\underline{y})$.

A5)

Claim 1: $pr_1|_{\Gamma(f)}$ is always bijective.

Proof: This is just a reformulation of the fact, that any map f maps any point in X to exactly one point in Y.

Claim 2: $pr_1|_{\Gamma(f)}$ is always continuous

Proof: $pr_1|_{\Gamma(f)}: \Gamma(f) \xrightarrow{i} X \times Y \xrightarrow{pr_1} X$

with i continuous by definition of the subspace topology and

pr_1 continuous by definition of the product topology.

Claim 3: $\forall f: X \to Y$ and $\forall U \subseteq X$: $(pr_1|_{\Gamma(f)})((U \times Y) \cap \Gamma(f))$ is open.

Proof: $(U \times Y) \cap \Gamma(f) = \{(x, f(x)) \in X \times Y \mid x \in U\}$

So: $(pr_1|_{\Gamma(f)})((U \times Y) \cap \Gamma(f)) = \{x \in X \mid x \in U\} = U$ indeed open.

Claim 4: $f: X \to Y$ is continuous if and only if $\forall V \subseteq Y$

$(pr_1|_{\Gamma(f)})((X \times V) \cap \Gamma(f))$ is open.

Proof: $(X \times V) \cap \Gamma(f) = \{(x, f(x)) \in X \times Y \mid f(x) \in V\} =$

$= \{(x, f(x)) \in X \times Y \mid x \in f^{-1}(V)\}$

$\Rightarrow (pr_1|_{\Gamma(f)})((X \times V) \cap \Gamma(f)) = \{x \in X \mid x \in f^{-1}(V)\} = f^{-1}(V)$

and the assertion follows.

By definition of the subspace and the product topology, the open sets $(U \times Y) \cap \Gamma(f)$ and $(X \times V) \cap \Gamma(f)$ (as above) form a subbasis for the topology on $\Gamma(f)$. Hence:

Claim 3 + 4 imply : $f: X \to Y$ continuous $\Leftrightarrow$ $pr_1|_{\Gamma(f)}$ open

with Claim 1 + 2 : $f: X \to Y$ continuous $\Leftrightarrow$ $pr_1|_{\Gamma(f)}$ homeomorphism.

A 6) a) $pr(U)$ is an open neighbourhood of p :

$p \in pr(U)$ and $U = pr^{-1}(pr(U))$ open because

$$U = \bigcup_{(x,y) \in U} B_{\varepsilon(x,y)}(x,y) \quad \text{with } \varepsilon(x,y) = \frac{1 - |x||y|}{|x| + |y| + 1}.$$

$pr(V)$ is not an open neighbourhood of p, because

$pr^{-1}(pr(V)) = V \cup (\mathbb{R} \times \{0\})$ is not open in $\mathbb{R}^2$.

b) One can just take

$$\mathcal{B} = \{pr(U) \mid U \subseteq \mathbb{R}^2 \text{ open}, \ \mathbb{R} \times \{0\} \subseteq U\}$$

Alternatively (and more explicitely):

$$\mathcal{B}' = \{\{(x,y) \mid |y| < f(x)\} \mid f: \mathbb{R} \to \mathbb{R} \text{ continuous}, f(x) > 0 \ \forall x\}$$

A6 c) Assume that $\tilde{\mathcal{B}} = \{U_i\}_{i \in \mathbb{Z}}$ is a countable neighbourhood basis of p. We construct now another open neighbourhood of p, which does not contain any of the U_i.

For this choose for each $i \in \mathbb{Z}$ some $a_i > 0$ s.th.

$$pr^{-1}(U_i) \cap (\{i\} \times \mathbb{R}) \not\subseteq (-a_i, a_i).$$

Now choose any continuous function $f: \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x) > 0 \ \forall x \in \mathbb{R} \quad \text{and} \quad f(i) = a_i \ \forall i \in \mathbb{Z}.$$

(e.g. by connecting the point (i, a_i) and $(i+1, a_{i+1})$ by linear functions)

This way we get an open subset

$$V = \{ (x,y) \in \mathbb{R}^2 \mid |y| < f(x) \} \subseteq \mathbb{R}^2$$

As $p \in pr(V)$ and $V = pr^{-1}(pr(V))$, ~~this~~ $pr(V)$ defines an open neighbourhood of p.

So it remains to check $U_i \not\subseteq pr(V)$ for any i. But by construction we may choose some $b_i \in \mathbb{R}$ with $|b_i| > a_i$ and $(i, b_i) \in U_i$. However $(i, b_i) \notin pr(V)$.

$\Rightarrow$ $pr(V)$ is an open neighbourhood of p not containing any U_i

$\Rightarrow$ p admits no countable neighbourhood basis

$\Rightarrow$ $\mathbb{R}^2/\sim$ is not first-countable.