Solutions to Sheet 3

A7) In all examples $\mathbb{R}$ and $\mathbb{R}^2$ are given the euclidean topology.

a) $M_1 = [0,1] \subseteq X_1 = \mathbb{R}$. Then $\partial M_1 = \{0,1\}$.

b) $M_2 = B_1((0,0)) \cup ([0,3] \times \{0\}) \cup B_1((3,0)) \subseteq X_2 = \mathbb{R}^2$

Then $\partial M_2 = B_1((0,0)) \cup B_1((3,0))$.

c) $M_3 = \mathbb{R} \setminus \{0\} \subseteq X_3 = \mathbb{R}$. Then $\partial M_3 = \{0\}$.

d) $M_4 = [0,1] \cup \{2\} \subseteq X_4 = \mathbb{R}$. Then $\mathring{M}_4 = (0,1)$.

e) $M_5 = S^1 \cap ((-\infty, \frac{1}{2}) \times \mathbb{R}) \subseteq \mathbb{R}^2$, $M_5' = S^1 \cap ((-\frac{1}{2}, \infty) \times \mathbb{R}) \subseteq \mathbb{R}^2$.

Then $M_5 \cap M_5' = S^1 \cap ((-\frac{1}{2}, \frac{1}{2}) \times \mathbb{R})$.

A8) Assume X is not connected. Then pick $U \subsetneq X$ open and closed (and non-empty) and set $V = X \setminus U$.

Case 1: $f(U) \cap f(V) \neq \emptyset$

Then pick $y \in f(U) \cap f(V) \subseteq Y$ and consider $f^{-1}(y)$.

By definition of the subspace topology, both $f^{-1}(y) \cap U$ and $f^{-1}(y) \cap V$ are open in $f^{-1}(y)$. Moreover

$$(f^{-1}(y) \cap U) \cup (f^{-1}(y) \cap V) = f^{-1}(y) \cap (U \cup V) = f^{-1}(y)$$

$$(f^{-1}(y) \cap U) \cap (f^{-1}(y) \cap V) = \emptyset$$

So $f^{-1}(y) \cong (f^{-1}(y) \cap U) + (f^{-1}(y) \cap V)$ is not connected

Case 2: $f(U) \cap f(V) = \emptyset$

This implies $V \cap f^{-1}(f(U)) = \emptyset = U \cap f^{-1}(f(V))$. Thus $U = f^{-1}(f(U))$ and $V = f^{-1}(f(V))$ and by definition of the quotient topology, $f(U) \subseteq Y$ and $f(V) \subseteq Y$ are open.

As obviously $f(U) \cup f(V) = Y$ and by assumption $f(U) \cap f(V) = \emptyset$,

$$Y = f(U) + f(V)$$

and Y is not connected.

Thus we get a contradiction in both cases.

A9) Assume $f: \mathbb{R} \longrightarrow \mathbb{R}^n$ is a homeomorphism and $n \geq 2$.

Then by definition of the subspace topology its restriction

$$\tilde{f} = f|_{\mathbb{R}\setminus\{0\}} : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}^n \setminus \{f(0)\}$$

is again bijective and continuous. As (via the same arguments) the same holds for $\tilde{f}^{-1}: \mathbb{R}^n \setminus \{f(0)\} \to \mathbb{R}\setminus\{0\}$, $\tilde{f}$ is a homeomorphism. Now:

- $\pi_0(\mathbb{R}\setminus\{0\}) = \{[-1],[1]\}$, because $\mathbb{R}\setminus\{0\} = (-\infty,0) + (0,\infty)$ and both $(-\infty,0)$ and $(0,\infty)$ are path-connected.
- $\pi_0(\mathbb{R}^n \setminus \{f(0)\}) = \{pt.\}$, because one can easily construct paths connecting any two points.

But any homeomorphism f induces a bijection $\pi_0(f)$.

Contradiction!