Solutions to Sheet 4

A 10,

i) X is (T1): If both points $x, y \in X$ lie in one of the two $\mathbb{R}$-pieces that were glued, they can be seperated because $\mathbb{R}$ is (T1) (even (T4)).

The only other case is $x = 0_1$, $y = 0_2$. Then

$(\mathbb{R}\setminus\{0\}) \cup \{0_1\}$ is open nbhd of x, not containing y

$(\mathbb{R}\setminus\{0\}) \cup \{0_2\}$ - - - - y, not containing x.

ii) X is not (T2): Let $x = 0_1$ and $y = 0_2$ and choose any open neighborhoods ~~of~~ U_x of x and U_y of y.

Then there are $\varepsilon_x, \varepsilon_y > 0$ s.th.

$$(-\varepsilon_x, \varepsilon_x)\setminus\{0\} \subseteq U_x \text{ and } (-\varepsilon_y, \varepsilon_y)\setminus\{0\} \subseteq U_y$$

So $(-\min(\varepsilon_x, \varepsilon_y), \min(\varepsilon_x, \varepsilon_y))\setminus\{0\} \subseteq U_x \cap U_y$ is non-empty. So X cannot be (T2).

iii) Y is (T2): $\mathbb{R}$ with the euclid. topology is (T2) and Y has a finer topology. So Y is (T2) as well.

iv) Y is not (T3): Let $x = \{0\}$ closed point and

$Z = \{\frac{1}{n}, n \in \mathbb{Z}, n > 0\} \subseteq Y$ a closed subset.

Assume we have open neighborhoods U_x of x and U_Z of Z with $U_x \cap U_Z = \emptyset$. $\mathcal{B}$ being a basis of, we may choose $\varepsilon > 0$ s.th.

$$(-\varepsilon, \varepsilon) \cap (\mathbb{R}\setminus Z) \subseteq U_x.$$

Then pick $n_0 \gg 0$ s.th. $\frac{1}{n_0} < \varepsilon$. As $\frac{1}{n_0} \in Z$ any open neighborhood of $\frac{1}{n_0}$ (like U_Z) contains some open interval around $\frac{1}{n_0}$, e.g. $(\frac{1}{n_0} - \varepsilon', \frac{1}{n_0} + \varepsilon')$ for some small $\varepsilon' > 0$

$\Rightarrow (-\varepsilon, \varepsilon) \cap (\frac{1}{n_0} - \varepsilon', \frac{1}{n_0} + \varepsilon')$ is an uncountable set

$\Rightarrow ((-\varepsilon, \varepsilon) \cap (\mathbb{R}\setminus Z)) \cap (\frac{1}{n_0} - \varepsilon', \frac{1}{n_0} + \varepsilon')$ is an uncountable set

$\Rightarrow U_x \cap U_Z$ is uncountable. Contradiction.

Together with

$(T4)+(T1) \Rightarrow (T3)+(T1) \Rightarrow (T2) \Rightarrow (T1) \Rightarrow (T0)$

we get:

X satisfies $(T0)$ and $(T1)$, but not $(T2), (T3), (T4)$

Y satisfies $(T0), (T1)$ and $(T2)$, but not $(T3), (T4)$.

A11)

(a) • $\mathcal{T}_0 = \{\text{complements of finite sets}\} \cup \{\emptyset\}$ is a topology on X. (argument as in exercise A1 (c)).

• ~~Let sets~~ $(X, \mathcal{T}_0)$ is $(T1)$ because for any $x, y \in X$ we may take the open neighbourhoods $X \setminus \{x\}$ and $X \setminus \{y\}$.

• Let now $\mathcal{T}$ be any topology on X s.th. $(X, \mathcal{T})$ is $(T1)$.

We show that $\mathcal{T}$ is finer than $\mathcal{T}_0$:

Pick any $x \in X$. Then for $y \neq x$ we may choose an open neighbourhood $U_y \in \mathcal{T}$ with $y \in U_y$, $x \notin U_y$. Then

$$X \setminus \{x\} = \bigcup_{y \neq x} U_y \in \mathcal{T}$$

Now if $Z \subseteq X$ is any finite set, then

$$X \setminus Z = \bigcap_{x \in Z} (X \setminus \{x\}) \in \mathcal{T}$$

$\Rightarrow \mathcal{T}_0 \subseteq \mathcal{T}$.

(b) Let $\{U_i, i \in I\}$ be some open cover of $(X, \mathcal{T}_0)$.

Pick any U_{i_0} (that is non-empty). Then choose for any of the finitely many points $x \in X \setminus U_{i_0}$ some $i_x \in I$ s.th. $x \in U_{i_x}$. Then $\{U_{i_0}\} \cup \{U_{i_x} \mid x \in X \setminus U_{i_0}\}$ is a finite subcover of X

$\Rightarrow X$ is compact.

(c) Assume $y, z \in Y$ are limit points of a sequence $(x_i)_{i \in \mathbb{N}}$. Then for any two open neighbourhoods $U_y, U_z \subseteq Y$, almost all x_i are contained in both. In particular $U_y \cap U_z \neq \emptyset$. So Y cannot be $(T2)$.

A 12)

(a) Let $Z \subseteq X \times Y$ be any closed subset. We have to see that $pr_1(Z) \subseteq X$ is closed or equivalently that $X \setminus pr_1(Z) \subseteq X$ is open. So pick any $x \in X \setminus pr_1(Z)$.

For any $y \in Y$, $(x,y) \notin Z$ and Z is closed. So we may choose some open $U_y \subseteq X$ and $V_y \subseteq Y$ s.th.

$$(x,y) \in U_y \times V_y \subseteq (X \times Y) \setminus Z.$$

By definition $\{V_y \mid y \in Y\}$ is an open cover of Y. So by compactness, we may find a finite subcover $\{V_{y_i} \mid i \in I\}$.

Now set $U = \bigcap_{i \in I} U_{y_i}$. Then:

$$U \times Y \subseteq \bigcup_{i \in I} (U_{y_i} \times V_{y_i}) \subseteq \bigcup_{y \in Y} (U_y \times V_y) \subseteq (X \times Y) \setminus Z.$$

In particular:

$$U \notin pr_1(Z)$$

is an open neighbourhood of x in $X \setminus pr_1(Z)$.

$\Rightarrow pr_1(Z)$ is closed $\Rightarrow pr_1$ is closed.

(b) By definition of the subspace topology, any closed subset of $\Gamma(f)$ is of the form $Z \cap \Gamma(f)$ for a closed subset $Z \subseteq X \times Y$. Hence if $\Gamma(f)$ is closed, any of its closed subsets will as well be closed in $X \times Y$, i.e. the inclusion

$$\iota : \Gamma(f) \hookrightarrow X \times Y$$ is a closed morphism.

As Y is compact, it follows by (a) that

$$\Gamma(f) \xhookrightarrow{\iota} X \times Y \xrightarrow{pr_1} X$$

is a closed morphism.

By exercise A5) it is automatically bijective and continuous. Hence it is a homeomorphism. So exercise A5) implies now that f is continuous.

(c) Let $X = Y = \mathbb{R}$. Then

$$Z = \{(x,y) \in \mathbb{R}^2 \mid x \cdot y = 1\} \subseteq \mathbb{R}^2$$

is closed, but $pr_1(Z) = \mathbb{R} \setminus \{0\}$ is not closed.

For part (b), consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It is clearly not continuous, but

$$\Gamma(f) = Z \cup \{(0,0)\}$$

is closed.