Solutions to Sheet 5

A 13)

(i) X is not compact: Consider $X \hookrightarrow \mathbb{R}^2 \xrightarrow{pr_1} \mathbb{R}$ the projection to the x-axis. It has image $[0, \infty)$, which is not compact. Hence X cannot be compact either.

(ii) Y is compact: Let us first show that Y is closed in $\mathbb{R}^2$. For this pick any point $x \in \mathbb{R} \setminus Y$ and let d be its distance to $(0,0)$. Then $B_{d/4}(x)$ does not intersect $\partial B_{\frac{1}{n}}((\frac{1}{n}, 0))$ for any $n \geq (\frac{d}{4})^{-1}$. As $\bigcup_{n < (\frac{d}{4})^{-1}} \partial B_{\frac{1}{n}}((\frac{1}{n}, 0))$ is closed (finite union!), x has the open neighborhood

$$B_{\frac{d}{4}}(x) \setminus \bigcup_{n < (\frac{d}{4})^{-1}} \partial B_{\frac{1}{n}}((\tfrac{1}{n}, 0)).$$

Moreover Y is contained in the compact closed ball $\overline{B_3((0,0))}$, hence Y is compact by Lemma 5.3.

As compactness is preserved under homeomorphisms, X and Y cannot be homeomorphic.

A 14)

Let us first assume that $B = \{p\}$ is just one point. Choose for any $x \in A$ open neighborhoods $x \in U_x \subseteq X$, $p \in V_x \subseteq X$ with $U_x \cap V_x = \emptyset$. Then $\{U_x\}_{x \in A}$ is an open cover of A, hence admits a finite open subcover $\{U_{x_i}\}_{i \in I}$. Then $U = \bigcup_i U_{x_i}$ is an open neighborhood of A and $V = \bigcap_i V_{x_i}$ is an open neighborhood of $B = \{p\}$ with $U \cap V = \emptyset$.

Consider now the general case: Then by the previous part we may choose for each $y \in B$ an open neighborhood $A \subseteq \tilde{U}_y \subseteq X$ and $y \in \tilde{V}_y \subseteq X$ with $\tilde{U}_y \cap \tilde{V}_y = \emptyset$. Then pick a finite cover $\{\tilde{V}_{y_j}\}_{j \in J}$ of B and set

$$\tilde{\mathcal{U}} = \bigcap_j \tilde{\mathcal{U}}_{y_j} \quad \text{and} \quad \tilde{V} = \bigcup_j \tilde{V}_{y_j}$$

These are open neighborhoods of A resp. B with $\tilde{\mathcal{U}} \cap \tilde{V} = \emptyset$, as desired.

A 15)

(a) $\iota : X \longrightarrow \iota(X)$ is certainly bijective and by definition of $\mathcal{T}$ on Y open. To prove that it is continuous, we have to see that ~~and~~ $X \cap (Y \setminus K)$ is open in X, when $K \subseteq X$ is compact. But by Lemma 5.4. K is closed (recall X is Hausdorff), and thus $X \cap (Y \setminus K) = X \setminus K$ is open in X.

(b) We first prove that Y is Hausdorff: As $X \subseteq Y$ is open and Hausdorff, part (a) implies that we may separate points in $X \subseteq Y$. So it remains to see that we may separate ∞ from any $x \in X$. As X is locally compact, choose $x \in \mathcal{U} \subseteq K \subseteq X$ with $\mathcal{U}$ open, K compact. Then we get a separation by $x \in \mathcal{U}$ and $\infty \in Y \setminus K$.

To show that Y is compact, choose any open cover $\{\mathcal{U}_i\}_{i \in I}$ of Y. As it covers ∞, pick one $\mathcal{U}_{i_0}$ of the form $\mathcal{U}_{i_0} = Y \setminus K$ for a compact K. We may now choose a finite subcover $\{\mathcal{U}_j\}_{j \in J}$ of K. Then $\{\mathcal{U}_j\}_{j \in J} \cup \{\mathcal{U}_{i_0}\}$ is the desired finite subcover of Y.

(c) Let $Y' = X \cup \{\infty'\}$. Then set

$$f : Y' \longrightarrow Y, \quad f|_X = \mathrm{id}_X \quad \text{and} \quad f(\infty') = \infty.$$

We check that f is continuous: If $\mathcal{U} \subseteq X \subseteq Y$ is open, then property (a) for Y' implies that $\mathcal{U} \subseteq \iota'(X)$ is open. As Y' is Hausdorff, $\iota'(X) \subseteq Y'$ is open, hence $\mathcal{U} \subseteq Y'$ is open. If $\mathcal{U} = Y \setminus K \subseteq Y$, then $f^{-1}(\mathcal{U}) = Y' \setminus K$. Again K is compact in a Hausdorff space, hence closed. $\Rightarrow f^{-1}(\mathcal{U})$ open.

f is obviously bijective, Y' compact by assumption and Y Hausdorff by part (b). So Corollary 5.8. implies that ~~Y' is~~ then f is a homeomorphism.

(d) $\mathbb{R}^2$ is Hausdorff and locally compact (using closed balls as compact neighborhoods), so it has a one-point compactification. By part (c), it remains to check properties (a) and (b) for $\mathbb{S}^2$:

property (a): Let $\mathbb{S}^2 = \partial B_1((0,0,1)) \subseteq \mathbb{R}^3$, $\infty = (0,0,2) \in \mathbb{S}^2$. and $\mathbb{R}^2 = \{(x,y,0) \mid x,y \in \mathbb{R}\} \subseteq \mathbb{R}^3$. Then define:

$$\iota : \mathbb{R}^2 \longrightarrow \mathbb{S}^2 \setminus \{\infty\}$$

$p \longmapsto$ second intersection of the line through p and ∞ with $\mathbb{S}^2$.

and its inverse $\iota^{-1} : \mathbb{S}^2 \setminus \{\infty\} \to \mathbb{R}^2$ the stereographic projection defined in the same way.

We omit checking continuity of ι and ι^{-1} here.

property (b): $\mathbb{S}^2$ is Hausdorff as a subspace of the Hausdorff space $\mathbb{R}^3$. It is compact as a closed bounded subset in $\mathbb{R}^3$ (which implies that any function $f: \mathbb{S}^2 \to \mathbb{R}$ is bounded).

$\Longrightarrow$ $\mathbb{S}^2$ is the one-point compactification of $\mathbb{R}^2$.