Solutions to Sheet 6

A16)

(a) Assume we have a totally ordered chain $(\{V_{j,n}\}_{j\in J_n})_n$ in $\mathcal{C}$. Then consider $\bigcup_n \{V_{j,n}\}_{j\in J_n}$. We want to show that this is again in $\mathcal{C}$. So assume the contrary, i.e. that it contains a finite subcover $\{V_{k,n_k}\}$ of X. Because it is finite, there exists some n with $n \geq n_k \ \forall k$, which implies

$\{V_{k,n_k}\} \subseteq \{V_{j,n}\}_{j\in J_n}$

i.e. already $\{V_{j,n}\}_{j\in J_n}$ has a finite subcover. Contradiction to $\{V_{j,n}\}_{j\in J_n} \in \mathcal{C}$.

So we may apply the Lemma of Zorn and get the maximal element.

(b) Let $\mathcal{M} = \{V_j\}_{j\in J}$ be a maximal element in $\mathcal{C}$, $x\in X$ some point. Pick $j_0 \in J$ s.th. $x \in V_{j_0}$. Then by the subbasis property, there are $U_1, \dots, U_n \in S$ s.th.

$x \in \bigcap_{i=1}^{n} U_i \subseteq V_{j_0}$.

Assume that $U_i \notin \mathcal{M}$ for all i. Then $\mathcal{M} \cup \{U_i\} \notin \mathcal{C}$ by maximality of $\mathcal{M}$. So we may choose a finite subcover $\mathcal{M}_i \subseteq \mathcal{M} \cup \{U_i\}$. But then

$$\bigcup_i (\mathcal{M}_i \setminus \{U_i\}) \cup \{V_{j_0}\}$$

is a finite subcover of $\mathcal{M}$. Contradiction.

Hence $U_i \in \mathcal{M}$ for at least one i.

(c) If X is non-compact, there is an open cover $\{U_i\}_{i\in I}$ without a finite subcover. So the set $\mathcal{C}$ is non-empty and by (a) we may choose a maximal element $\mathcal{M}$ in $\mathcal{C}$. Now part (b) ensures that $\mathcal{M} \cap S$ is still a cover of X. However it does not contain a finite subcover, because this would be a finite subcover of $\mathcal{M}$ as well.

(d) On $\prod_{i\in I} X_i$ we use the subbasis $S = \{pr_i^{-1}(U_i) \mid U_i \subseteq X_i \text{ open}, i\in I\}$.

Assume $\{V_j\}_{j\in J} \subseteq S$ is a cover without a finite subcover.
Then $\{V_j\}_{j\in J} = \bigcup_{i\in I} \{V_{j_i}\}_{j_i\in J_i}$, where all V_{j_i} are of the form $pr_i^{-1}(U_i)$ for this index i.
Then $\{pr_i(V_{j_i})\}_{j_i\in J_i}$ cannot be a cover of X_i, because otherwise it would contain a finite subcover, which then defines an finite subcover of $\{V_j\}$.
So we may pick $x_i \in X_i$ with $x_i \notin pr_i(V_{j_i})\ \forall_{j_i} \in J_i$.
But then by construction the point $(x_i) \in \prod_i X_i$ does not lie in any of the V_j. Contradiction.

A17)

(a) $f \in M(C_X \times C_Y, U)$

$\iff f \in Hom(X\times Y, Z)$ s.th. $f(C_X \times C_Y) \subseteq U$

$\iff f \in Hom(X\times Y, Z)$ s.th. $\forall x \in C_X \quad f(\{x\}\times C_Y) \subseteq U$.

$\iff f \in Hom(X, Hom(Y,Z))$ s.th. $\forall x \in C_X \quad (f(x))(C_Y) \subseteq U$

$\iff f \in M(C_X, M(C_Y, U))$.

As the adjunction map $Hom(X\times Y, Z) \xrightarrow{\sim} Hom(X, Hom(Y,Z))$ is a bijection, this shows the assertion.

(b) It suffices to see that for every $C \subseteq X$ compact and every $V \subseteq T$ open, $M(C, V)$ is open in the topology generated by $\{M(C_X, U) \mid C_X \subseteq X \text{ compact}, U \in B\}$.
So pick any $f \in M(C, V)$ and write $V = \bigcup_{i\in I} U_i$ for some open sets $U_i \in B$. For a point $x \in C$ we may pick $i \in I$ s.th. $f(x) \in U_i$. Then $x \in f^{-1}(U_i) \cap C \subseteq C$ is an open neighborhood of x in C.

Using the statement in the remark, we may find a compact neighborhood A_x of x inside $f^{-1}(U_i) \cap C$.

In particular we have $f \in M(A_x, U_i)$.

We wish to show that a finite number of the A_x cover C.

For this consider their interior $\mathring{A}_x \subseteq C$ as a subset of C.

Warning: The A_x are in general not neighborhoods of x in X, but only neighborhoods of x in C! So $\mathring{A}_x$ is not the largest subset of A_x, which is open in X, but the largest subset, which is open in C.

By definition $x \in \mathring{A}_x$, so the $\{\mathring{A}_x\}_{x \in C}$ cover C. By compactness, there ~~are~~ is a finite subcover $\{\mathring{A}_{x_j}\}_j$. Then $\{A_{x_j}\}$ covers C as well. Then

$$f \in \bigcap_j M(A_{x_j}, U_{i_j}) = \{f \in \mathrm{Hom}(X,T) \mid f(A_{x_j}) \subseteq U_{i_j} \ \forall j\}$$
$$\subseteq \{f \in \mathrm{Hom}(X,T) \mid f(\bigcup_j A_{x_j}) \subseteq \bigcup_j U_{i_j}\}$$
$$\subseteq \{f \in \mathrm{Hom}(X,T) \mid f(C) \subseteq V\} = M(C,V)$$

$\Longrightarrow$ We have found the desired open neighborhood of f in $M(C,V)$.

(c) All the sets $\bigcap_i U_i$ for $U_i \in S$ form a basis for the topology on T, so it suffices by part (b) to show that $M(C_x, \bigcap_i U_i)$ is open in the topology generated by $\{M(C_x, U) \mid C_x \text{ compact}, U \in S\}$.

But:

$$M(C_x, \bigcap_i U_i) = \{f \in \mathrm{Hom}(X,T) \mid f(C_x) \subseteq \bigcap_i U_i\} =$$
$$= \{f \in \mathrm{Hom}(X,T) \mid f(C_x) \subseteq U_i \ \forall i\} =$$
$$= \bigcap_i \{f \in \mathrm{Hom}(X,T) \mid f(C_x) \subseteq U_i\} =$$
$$= \bigcap_i M(C_x, U_i)$$

showing precisely this.

(d) We already know that

$$ad : Hom(X \times Y, Z) \longrightarrow Hom(X, Hom(Y, Z))$$

is bijective. So it remains to see:

- ad is open: We may check this on the subbasis ⊛ $\{M(C_X \times C_Y, U)\}$. But by part (a) $ad(M(C_X \times C_Y, U)) = M(C_X, M(C_Y, U))$ is indeed open.
- ad is continuous: Again we may check this on a subbasis. Part (c) for $T = Hom(Y, Z)$ implies, that $\{M(C_X, M(C_Y, U)) \mid C_X, C_Y \text{ compact}, U \text{ open}\}$ is a subbasis of $Hom(X, Hom(Y, Z))$. So again by part (a):
$$ad^{-1}(M(C_X, M(C_Y, U))) = M(C_X \times C_Y, U)$$
and ad is continuous.

⊛ We still have to check that $\{M(C_X \times C_Y, U)\}$ is a subbasis. This is only true under the additional assumption:

X is locally compact.

If so consider $C \subseteq X \times Y$ any compact, $f \in M(C, U)$ some element. Then $f^{-1}(U) \subseteq X \times Y$ is an open neighborhood of C, hence can be covered by $\{U_{X,i} \times U_{Y,i}\}$. By local compactness of X and Y, we may find $\{C_{X,j} \times C_{Y,j}\}$ with compacts $C_{X,j}, C_{Y,j}$ s.th. their interiors cover C and every $C_{X,j} \times C_{Y,j}$ is contained in some $U_{X,i} \times U_{Y,i}$. By compactness of C, we can cover it by finitely many $\{C_{X,j} \times C_{Y,j}\}_{j \in J}$. Then

$$f \in \bigcap_{j \in J} M(C_{X,j} \times C_{Y,j}, U) \subseteq M(C, U)$$

giving the desired open neighborhood.