Solutions to Sheet 7

A 18)

(a) Consider the following diagram

$$\begin{tikzcd} U(n) \arrow[rr, hook] & & GL_n(\mathbb{C}) \\ U(n-k)\times U(k) \arrow[u, hook] \arrow[r, hook] & GL_{n-k}(\mathbb{C})\times GL_k(\mathbb{C}) \arrow[r, hook] & P_k(\mathbb{C}) \arrow[u, hook] \end{tikzcd}$$

All inclusions are by definition continuous. Thus we get a continuous morphism

$$Gr_k(\mathbb{C}^n) \cong U(n)/_{U(n-k)\times U(k)} \longrightarrow GL_n(\mathbb{C})/_{P_k(\mathbb{C})}.$$

To show bijectivity, we construct the inverse (as a map between sets; not as a continuous morphism): Consider

$$GL_n(\mathbb{C}) \longrightarrow Gr_k(\mathbb{C}^n) \quad , \quad f \longmapsto \langle f(e_1),\dots,f(e_k)\rangle \subseteq \mathbb{C}^n$$

(where $\{e_1,\dots,e_n\}$ is the standard basis of $\mathbb{C}^n$).

This is surjective (cf. linear algebra) and with

$$P_k(\mathbb{C}) = \{ f\in GL_n(\mathbb{C}) \mid f(\langle e_1,\dots,e_k\rangle) = \langle e_1,\dots,e_k\rangle \}$$

it is constant on $P_k(\mathbb{C})$-cosets. Hence we get a map

$$GL_n(\mathbb{C})/_{P_k(\mathbb{C})} \longrightarrow\!\!\!\!\!\rightarrow Gr_k(\mathbb{C}^n)$$

Plugging in the definitions, one sees that it is a (set-theoretic) inverse of $Gr_k(\mathbb{C}^n) \longrightarrow GL_n(\mathbb{C})/_{P_k(\mathbb{C})}$ from above.

As $Gr_k(\mathbb{C}^n)$ is compact, $GL_n(\mathbb{C})/_{P_k(\mathbb{C})}$ is Hausdorff (because $P_k(\mathbb{C})\subseteq GL_n(\mathbb{C})$ closed), we get a homeomorphism

$$Gr_k(\mathbb{C}^n) \overset{\approx}{=} GL_n(\mathbb{C})/_{P_k(\mathbb{C})}.$$

(b) Consider the action (on a set, not a topol. space):

$$\rho: GL_n(\mathbb{C}) \longrightarrow \mathrm{Aut}(Fl(\mathbb{C}^n))$$
$$f \longmapsto \big((V_1\subseteq\dots\subseteq V_n) \longmapsto (f(V_1)\subseteq\dots\subseteq f(V_n))\big)$$

and its restriction

$$\rho_U: U(n) \hookrightarrow GL_n(\mathbb{C}) \xrightarrow{\rho} \mathrm{Aut}(Fl(\mathbb{C}^n)).$$

Fix $x = (\langle e_1\rangle \subseteq \langle e_1,e_2\rangle \subseteq \dots \subseteq \langle e_1,e_2,\dots,e_n\rangle) \in Fl(\mathbb{C}^n)$.

Claim 1: SU acts transitively on $Fl(\mathbb{C}^n)$

Let $(V_1 \subseteq \ldots \subseteq V_n) \in Fl(\mathbb{C}^n)$ be a flag. Then we may inductively construct ~~...~~ an orthonormal basis $\{v_1, \ldots, v_n\}$ s.th. $V_i = \langle v_1, v_2, \ldots, v_i \rangle$ for all i. Let $g \in U(n)$ be the base-change matrix from $\{e_1, \ldots, e_n\}$ to $\{v_1, \ldots, v_n\}$. Then by construction: $SU(g)(x) = (V_1 \subseteq \ldots \subseteq V_n) \in Fl(\mathbb{C}^n)$.

Claim 2: The stabilizer of x in $GL_n(\mathbb{C})$ is $B(\mathbb{C})$.

$$GL_n(\mathbb{C})_x = \{ f \in GL_n(\mathbb{C}) \mid f(\langle e_1, e_2, \ldots, e_i \rangle) = \langle e_1, e_2, \ldots, e_i \rangle \, \forall i \}$$
$$= \{ f \in GL_n(\mathbb{C}) \mid f(e_i) \in \langle e_1, e_2, \ldots, e_i \rangle \; \forall i \}$$
$$= B(\mathbb{C}).$$

By claim 1, $GL_n(\mathbb{C})$ acts transitively as well, so
$$GL_n(\mathbb{C}) / GL_n(\mathbb{C})_x = GL_n(\mathbb{C}) / B(\mathbb{C}) \cong Fl(\mathbb{C}^n)$$
as sets.

Moreover we have a bijection of sets
$$U(n) / U(n)_x \cong GL_n(\mathbb{C}) / GL_n(\mathbb{C})_x = GL_n(\mathbb{C}) / B(\mathbb{C}).$$
In particular
$$U(n) \hookrightarrow GL_n(\mathbb{C}) \longrightarrow GL_n(\mathbb{C}) / B(\mathbb{C})$$
is continuous and surjective. So compactness of $U(n)$ implies compactness of $GL_n(\mathbb{C}) / B(\mathbb{C})$.

A 19)

(a) i is a homeomorphism, hence preserves closures, i.e.
$$i(\overline{H}) = \overline{i(H)} = \overline{H}.$$
For any $g \in G$, translation wrt. g is a homeomorphism. Hence for $g \in H$, $\overline{H}g$ is closed in G and contains $Hg = H$.
$$\Rightarrow \overline{H}g = \overline{H} \quad \forall g \in H$$
Now for $\bar{g} \in \overline{H}$, $\bar{g}\overline{H}$ is closed and contains $H = \bar{g}(\bar{g}^{-1}H)$
$$\Rightarrow \bar{g}\overline{H} = \overline{H} \quad \forall \bar{g} \in \overline{H}.$$

(b) Conjugation is a homeomorphism, so for any $x \in G$:

$x \overline{H} x^{-1}$ is closed and contains $x H x^{-1} = H$

$\Rightarrow x \overline{H} x^{-1} = \overline{H}$

$\rightarrow \overline{H} \subseteq G$ normal.

(c) Consider the continuous map

$c : G \times G \longrightarrow G \ , \ (g,h) \mapsto g \cdot h \cdot g^{-1} \cdot h^{-1}$

Then by assumption $H \times H \subseteq c^{-1}(e\}$ and $c^{-1}(e)$ is closed

$\Rightarrow \overline{H \times H} \subseteq c^{-1}(e)$

But $\overline{H \times H} = \overline{H} \times \overline{H}$: Indeed if $x \in \overline{H} \times \overline{H} \setminus \overline{H \times H}$, there is an open nbhd $x \in U \times V \subseteq G \times G \setminus H \times H$. So either ~~[illegible]~~ $U \subseteq G \setminus H$ or $V \subseteq G \setminus H$ contradicting $x \in \overline{H} \times \overline{H}$.

$\Rightarrow \overline{H} \times \overline{H} \subseteq c^{-1}(e)$ i.e. $\overline{H}$ abelian.

A20

(a) Let $x, y \in G$. Then by (T0) we may find wlog. an open neighborhood $x \in V \subseteq G \setminus \{y\}$. Let $V^{-1} = i(V)$ open. Then $x \cdot V^{-1} \cdot y \subseteq G$ is open as well with

- $y = x \cdot x^{-1} \cdot y \in x \cdot V^{-1} \cdot y$
- $y \notin V \Rightarrow V^{-1} \cdot y \not\ni e \Rightarrow x \cdot V^{-1} \cdot y \not\ni x$

(T1) $\Rightarrow$ (T2) cf. lecture

$\Rightarrow x \cdot V^{-1} \cdot y$ is open neighborhood of y in $G \setminus \{x\}$.

(b) Let $g \in \overline{U}$. Then gU and U are open subsets in $\overline{U}$, hence $gU \cap U \neq \emptyset$,

i.e. $\exists u_1, u_2 \in U$ with $g \cdot u_1 = u_2$

$\Rightarrow g = u_1^{-1} \cdot u_2 \in U^{-1} \cdot U$.

(c) Let $x \in G$ and $A \subseteq G \setminus \{x\}$ closed in G. As translations are homeos, assume wlog $x = e$ is the identity.

Now $m^{-1}(G \setminus A) \subseteq G \times G$ is open and contains (e,e).

Hence there are open neighborhoods V_1, V_2 of e with $V_1 \times V_2 \subseteq m^{-1}(G \setminus A)$.

Define $\mathcal{U} = V_1 \cap V_2 \cap i(V_1) \cap i(V_2) = V_1 \cap V_2 \cap V_1^{-1} \cap V_2^{-1}$.

Then by construction $\mathcal{U} = i(\mathcal{U}) = \mathcal{U}^{-1}$ and

$$\mathcal{U} \cdot \mathcal{U} = m(\mathcal{U} \times \mathcal{U}) \subseteq m(V_1 \times V_2) \subseteq G \setminus A.$$

So by part (b):

$$\overline{\mathcal{U}} \subseteq \mathcal{U}^{-1} \cdot \mathcal{U} = \mathcal{U} \cdot \mathcal{U} \subseteq G \setminus A.$$

$\Longrightarrow$ $e \in \mathcal{U}$ and $A \subseteq G \setminus \overline{\mathcal{U}}$ disjoint open neighborhoods.