Solutions for Sheet 8

A 21) Denote the homotopies by

$$H_f : I \times X \longrightarrow X' \quad \text{from } f_1 \text{ to } f_2$$
$$H_g : I \times Y \longrightarrow Y' \quad \text{from } g_1 \text{ to } g_2$$

a) $H_f + H_g : I \times (X+Y) \cong (I \times X) + (I \times Y) \longrightarrow X' + Y'$

is a homotopy from $f_1 + g_1$ to $f_2 + g_2$.

b) Consider the map

$$H_{f \times g} : I \times (X \times Y) \xrightarrow{\Delta_I \times id_{X \times Y}} (I \times I) \times (X \times Y) \cong (I \times X) \times (I \times Y) \longrightarrow$$
$$\xrightarrow{H_f \times H_g} X' \times Y'$$

where $\Delta_I : I \longrightarrow I \times I$ is the diagonal morphism.

Then $H_{f \times g}$ is the desired homotopy

c) By adjunction, we have paths

$$H_f^\# : I \longrightarrow Hom(X, X')$$
$$H_g^\# : I \longrightarrow Hom(Y, Y')$$

Then consider:

$$H : I \times Hom(X', Y) \xrightarrow{\Delta_I \times id} (I \times I) \times Hom(X', Y) \cong$$
$$\cong I \times Hom(X', Y) \times I \xrightarrow{H_f^\# \times id \times H_g^\#}$$
$$\longrightarrow Hom(X, X') \times Hom(X', Y) \times Hom(Y, Y') \longrightarrow$$
$$\xrightarrow{comp} Hom(X, Y')$$

with the composition map being continuous, because all spaces are locally compact.

Now for any $h \in Hom(X', Y)$:

~~$H(\Delta f) = comp(H_f^\#(0), f_T$~~

$$H(0, h) = comp(H_f^\#(0), h, H_g^\#(0)) =$$
$$= comp(f_1, h, g_1) = g_1 \circ h \circ f_1$$

and $H(1, h) = \ldots = g_2 \circ h \circ f_2$.

$\Rightarrow H$ is the desired homotopy.

A22)

(a) Let $f: SL_2(\mathbb{R}) \hookrightarrow GL_2(\mathbb{R})^+$ the canonical inclusion

$$g: GL_2(\mathbb{R})^+ \longrightarrow SL_2(\mathbb{R})$$
$$A \longmapsto \begin{pmatrix} \det A^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot A$$

Obviously $g \circ f = id_{SL_2(\mathbb{R})}$, so $[g \circ f] = [id]$. Moreover $[f \circ g] = [id_{GL_2(\mathbb{R})^+}]$ holds, due to the homotopy

$$H: \cancel{I} \, I \times GL_2(\mathbb{R})^+ \longrightarrow GL_2(\mathbb{R})^+$$
$$(t, A) \longmapsto \begin{pmatrix} (1-t) + t \cdot \det A^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot A$$

which is well-defined, because

$$(1-t) + t \cdot \det A^{-1} \neq 0 \quad \forall t \in [0,1].$$

(b) Let $f: SO_2(\mathbb{R}) \hookrightarrow GL_2(\mathbb{R})^+$ the canonical inclusion

$$g: GL_2(\mathbb{R})^+ \longrightarrow SO_2(\mathbb{R})$$
$$A = (v, v') \longmapsto \left(\frac{v}{\|v\|}, \frac{v' - \langle v, v' \rangle v}{\|v' - \langle v, v' \rangle v\|} \right)$$

i.e. we apply Gram-Schmidt to the column vectors of A.

Again $g \circ f = id_{SO_2(\mathbb{R})}$ is obvious. We claim that we get a homotopy from $f \circ g$ to $id_{GL_2(\mathbb{R})^+}$ as follows:

$$H: I \times GL_2(\mathbb{R})^+ \longrightarrow GL_2(\mathbb{R})^+$$
$$(t, A = (v, v')) \longmapsto \left((1-t)v + t\frac{v}{\|v\|}, (1-t)v' + t\frac{v' - \langle v, v' \rangle v}{\|v' - \langle v, v' \rangle v\|} \right).$$

LA tells us easily, that H is a continuous map to $GL_2(\mathbb{R})$.

To see that H maps into $GL_2(\mathbb{R})^+$, we may argue as follows:

If $GL_2(\mathbb{R})^- = \{A \in GL_2(\mathbb{R}) \mid \det(A) < 0\}$, then

$$GL_2(\mathbb{R}) = GL_2(\mathbb{R})^+ \, \cancel{\sqcup} + GL_2(\mathbb{R})^-$$

because $\det: GL_2(\mathbb{R}) \to \mathbb{R} \setminus \{0\} \; \cancel{\text{and}} = (-\infty, 0) + (0, \infty)$ and $GL_2(\mathbb{R})^\pm$ are the inverse images of these components.

Now for any $A \in GL_2(\mathbb{R})^+$, $H|_{I \times \{A\}}$ is connected with $H(\{0\}, A) = A \in GL_2(\mathbb{R})^+$

$\Rightarrow H|_{I \times \{A\}}$ has image inside $GL_2(\mathbb{R})^+$ (by connectedness)

$\Rightarrow H$ is well-defined.

Obviously $H|_{\{0\} \times GL_2(\mathbb{R})^+} = \mathrm{id}$ and $H|_{\{1\} \times GL_2(\mathbb{R})^+} = f \circ g$.

(c) In part (b) we have constructed a homotopy equivalence

$$H: GL_2(\mathbb{R})^+ \longrightarrow SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} \sin\varphi & \cos\varphi \\ -\cos\varphi & \sin\varphi \end{pmatrix}, \varphi \in \mathbb{R} \right\} \overset{\varphi}{=}$$

$$\cong \mathbb{R}/_{2\pi\mathbb{Z}} \cong S^1.$$

Now $i: GL_2(\mathbb{R})^+ \longrightarrow GL_2(\mathbb{R})^-$, $A \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A$ is a homeomorphism, so we get a homotopy equivalence:

$$GL_2(\mathbb{R}) = GL_2(\mathbb{R})^+ + GL_2(\mathbb{R})^- \xrightarrow{H + H \circ i} S^1 + S^1.$$

as desired.