Solutions to Sheet 9

A 24) Let $pr_1 : X \times Y \to X$, $pr_2 : X \times Y \to Y$ be the canonical projections. Then we have a map of groups

$$(pr_{1*}, pr_{2*}) : \pi_1(X \times Y) \longrightarrow \pi_1(X) \times \pi_1(Y)$$

We claim that it has an inverse given by

$$f : \pi_1(X) \times \pi_1(Y) \longrightarrow \pi_1(X \times Y)$$
$$(\gamma \;,\; \gamma') \longmapsto (\gamma . \gamma') : I \longrightarrow X \times Y$$
$$t \longmapsto (\gamma(t), \gamma'(t))$$

a) f is well-defined.

If γ_1, γ_2 are two representatives of an element in $\pi_1(X)$, then there is a homotopy $H : I \times I \to X$ from γ_1 to γ_2.

Then $f(\gamma_1, \gamma') \simeq f(\gamma_2, \gamma')$ are homotopic via

$$\tilde{H} : I \times I \longrightarrow X \times Y$$
$$(s, t) \longmapsto (H(s,t), \gamma'(t))$$

Indeed: $\tilde{H}|_{\{0\} \times I}(t) = (H(0,t), \gamma'(t)) =$

$= (\gamma_1(t), \gamma'(t)) = f(\gamma_1 . \gamma')(t)$

and similarly: $\tilde{H}|_{\{1\} \times I} = f(\gamma_2, \gamma')$.

Similarly for two representatives in $\pi_1(Y)$.

b) f is an inverse to (pr_{1*}, pr_{2*})

$(pr_{1*}, pr_{2*}) \circ f(\gamma, \gamma')(t) =$

$= (pr_1(f(\gamma, \gamma')(t)), pr_2(f(\gamma, \gamma')(t))) =$

$= (pr_1(\gamma(t), \gamma'(t)), pr_2(\gamma(t), \gamma'(t))) =$

$= (\gamma(t), \gamma'(t)) \qquad \forall \gamma \in \pi_1(X), \gamma' \in \pi_1(Y), t \in I$

$\Rightarrow (pr_{1*}, pr_{2*}) \circ f = id$ already without taking equivalence classes of homotopic paths.

In the same way $f \circ (pr_{1*}, pr_{2*}) = id$ already without taking equivalence classes.

A25) a) If $\gamma_1 \simeq \gamma_2$ are homotopic via $H: I \times I \to G$ and $\gamma' \in \Omega$ any other path, then consider

$$H \cdot \gamma' : I \times I \longrightarrow G$$
$$(s,t) \longmapsto H(s,t) \cdot \gamma'(t)$$

It satisfies $H \cdot \gamma'|_{\{0\} \times I} = \gamma_1 \diamond \gamma'$ and $H \cdot \gamma'|_{\{1\} \times I} = \gamma_2 \diamond \gamma'$.

$\Rightarrow \gamma_1 \diamond \gamma' \simeq \gamma_2 \diamond \gamma'$ are homotopic.

Analogous for varying the second path within its homotopy class.

$\Rightarrow$ Get a well-defined map

$$\diamond : \pi_1(G) \times \pi_1(G) \longrightarrow \pi_1(G).$$

b) We compute

$$(\gamma c_e) \diamond (c_e \gamma')(t) = (\gamma c_e)(t) \cdot (c_e \gamma')(t) =$$
$$= \begin{cases} \gamma(2t) \cdot c_e(2t) = \gamma(2t) & \text{if } t \in [0, \tfrac{1}{2}] \\ c_e(2t-1) \cdot \gamma'(2t-1) = \gamma'(2t-1) & \text{if } t \in (\tfrac{1}{2}, 1]. \end{cases}$$

$\Rightarrow (\gamma c_e) \diamond (c_e \gamma') = \gamma * \gamma'$ in $\pi_1(G)$.

As $\gamma \simeq \gamma c_e$ and $\gamma' \simeq c_e \gamma'$ we get by a)

$$\gamma \diamond \gamma' = (\gamma c_e) \diamond (c_e \gamma') = \gamma * \gamma'.$$

c) A direct computation as in b) gives

$$(c_e \gamma) \diamond (\gamma' c_e) = \gamma' * \gamma$$

So: $\gamma * \gamma' = (\gamma c_e) \diamond (c_e \gamma') = \gamma \diamond \gamma' = (c_e \gamma) \diamond (\gamma' c_e) = \gamma' * \gamma$

for any $\gamma, \gamma' \in \pi_1(G)$

$\Rightarrow \pi_1(G)$ is commutative.

A 26)

i) Not a covering.

Let $\mathcal{U} \subseteq [0,3]$ be any open neighborhood of 2. Then there is $1 > \varepsilon > 0$ with $\{2+\varepsilon, 2-\varepsilon\} \in \mathcal{U}$, satisfying $f(2+\varepsilon) = \varepsilon = f(2-\varepsilon)$.

$\Rightarrow$ f not injective on any open neighborhood of 2.

$\Rightarrow$ f not a local homeomorphism

$\Rightarrow$ f not a covering.

ii) q is a covering.

Let $A \subseteq S^1$ be a closed arc of lenth at most π (i.e. A does not contain two antipodal points). Let $-A = \{-x \mid x \in A\}$ be the arc opposite to A.

Let $B = q(A \times I) \subseteq M$. Then

$q^{-1}(B) = (A \times I) \amalg (-A \times I)$

with (path) connected components $A \times I$ and $-A \times I$.

Now $q : A \times I \longrightarrow B$ is continuous, bijective, $A \times I$ compact and B Hausdorff. So q is a homeomorphism.

Similarly $q : -A \times I \longrightarrow B$ is a homeomorphism.

$\Longrightarrow$ For all $B = q(A \times I)$, $q : q^{-1}(B) \longrightarrow B$ is a homeo. when restricted to pathconnected components on the source.

As any point in M has a neighborhood of the form $q(A \times I)$, this shows that q is a covering.

iii) Not a covering (though it is a local homeo).

The $\mathcal{U}_\varepsilon = \{e^{2\pi i t}, t \in (-\varepsilon, \varepsilon)\} \subseteq S^1$ ⊛ form a neighborhood basis of $1 \in S^1$. We claim that for no $\mathcal{U}_\varepsilon$ the map

$\exp : \exp^{-1}(\mathcal{U}_\varepsilon) \longrightarrow \mathcal{U}_\varepsilon$

is an homeomorphism on each connected component.

⊛ for $0 < \varepsilon < \frac{1}{2}$

So consider one U_ε (with $\varepsilon < \frac{1}{2}$). Then

$$\exp^{-1}(U_\varepsilon) = (0, \varepsilon) + (1-\varepsilon, 1+\varepsilon) + (2-\varepsilon, 2+\varepsilon) + \ldots$$

with connected components $(0, \varepsilon)$ and $(n-\varepsilon, n+\varepsilon)$ for all $n \in \mathbb{Z}_{\geq 1}$. But

$$\exp|_{(0,\varepsilon)} : (0, \varepsilon) \longrightarrow U_\varepsilon$$

is not a homeomorphism.

$\Rightarrow \exp : (0, \infty) \longrightarrow S^1$ does not satisfy the conditions for a covering morphism around the point $1 \in S^1$.