Technische Universität München
Zentrum Mathematik

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WS 2015/16
Sheet 1

Topology (MA 3241)

Exercises (to be handed in Thursday, 22.10.2015, before the lecture)

A 1 Are the following objects topological spaces or not? Give a brief explanation.

(a) $X_1 = \mathbb{R}$ and $\mathcal{T}_1 = \{U \subset X_1 \,|\, U \text{ is a union of open intervals}\}$

(b) $X_2 = \mathbb{R}$ and $\mathcal{T}_2 = \{U \subset X_2 \,|\, U \text{ is a finite set}\} \cup \{X_2\}$

(c) $X_3 = \mathbb{R}$ and $\mathcal{T}_3 = \{U \subset X_3 \,|\, X_3 \setminus U \text{ is a finite set}\} \cup \{\emptyset\}$

(d) $X_4 = \mathbb{R} \times \mathbb{R}$ and $\mathcal{T}_4 = \{U \times U' \subset X_4 \,|\, U \text{ and } U' \subset \mathbb{R} \text{ are unions of open intervals}\}$

A 2 (a) Let X be a set and $\{\mathcal{T}_i\}_{i \in I}$ a family of topologies on X (indexed by some set I). Show that the intersection $\bigcap_{i \in I} \mathcal{T}_i = \{U \subset X \,|\, U \in \mathcal{T}_i \;\; \forall i \in I\}$ is another topology on X.

(b) Find an example of a set X and two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X, such that their union $\mathcal{T}_1 \cup \mathcal{T}_2 = \{U \subset X \,|\, U \in \mathcal{T}_1 \text{ or } U \in \mathcal{T}_2\}$ is *not* a topology on X.

(c) Let X be a set, $\mathcal{T}_1$ and $\mathcal{T}_2$ two topologies on X. Show that there is a unique topology $\mathcal{T}$ on X with $\mathcal{T}_1 \cup \mathcal{T}_2 \subset \mathcal{T}$, such that any other topology $\mathcal{T}'$ with $\mathcal{T}_1 \cup \mathcal{T}_2 \subset \mathcal{T}'$ is finer than $\mathcal{T}$.

Hint for (c): Consider the intersection of all topologies containing $\mathcal{T}_1 \cup \mathcal{T}_2$. Alternatively you can show that $\mathcal{T}$ is the unique topology for which $\mathcal{T}_1 \cup \mathcal{T}_2$ is a subbasis.

A 3 Let $X = \mathbb{R}$ and $\mathcal{T} = \{(-r, r) \subset \mathbb{R} \,|\, r \in \mathbb{R}, r > 0\} \cup \{\mathbb{R}, \emptyset\}$ a topology on X. Moreover let

$$f : (X, \mathcal{T}) \to (X, \mathcal{T}) \quad , \quad f(x) = \begin{cases} -x & \text{if } |x| < 1 \\ 2x & \text{if } |x| \geq 1 \end{cases}$$

(a) Is f continuous? Is f open? Is f closed?

(b) Show that $(X, \mathcal{T})$ is second-countable.

(c) Find the set of all limits of the sequence $(x_n)_{n \in \mathbb{N}}$ given by $x_n = 1 + \frac{1}{n} \in X$.

(d) Find the interior $\mathring{M}$ and the boundary ∂M of the set $M = [-1, 2] \subset X$.

Solutions to the exercises will be available from October 22, 2015 on, at

`https://www-m11.ma.tum.de/viehmann/topology/`

The **Notenbonus** can be obtained by making for at least 70% of all exercises a serious effort to solve it.