Technische Universität München
Zentrum Mathematik

WS 2015/16
Sheet 10

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Topology (MA 3241)

Exercises (to be handed in Thursday, 7.1.2016, before the lecture)

A 27 Let X and Y be path-connected and locally path-connected spaces, and $p : (X, x_0) \to (Y, y_0)$ be a covering. Define the group of deck transforms as

$$\mathrm{Aut}(X \xrightarrow{p} Y) = \{f : X \to X \text{ homeomorphism} \mid p \circ f = p : X \to Y\}$$

(a) Show that any element $f \in \mathrm{Aut}(X \xrightarrow{p} Y)$ is uniquely determined by fixing $f(x_0) \in p^{-1}(y_0)$.

(b) Let $\alpha' : I \to X$ be a path from $x_0 = \alpha'(0)$ to $x_1 = \alpha'(1)$. Then the loop $p \circ \alpha' : I \to Y$ defines a class $[\alpha] \in \pi_1(Y, y_0)$. Prove that

$$p_*(\pi_1(X, x_1)) = [\alpha^{-1}] * p_*(\pi_1(X, x_0)) * [\alpha]$$

inside $\pi_1(Y, y_0)$.

(c) Show that $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(Y, y_0)$ if and only if $\mathrm{Aut}(X \xrightarrow{p} Y)$ acts transitively on $p^{-1}(y_0)$.

(d) Apply the properties in part (c) hold. Prove that there is a surjective morphism of groups $\pi_1(Y, y_0) \to \mathrm{Aut}(X \xrightarrow{p} Y)$ with kernel isomorphic to $\pi_1(X, x_0)$.

Remark: The last statement is of particular interest, when $\pi_1(X, x_0)$ is trivial. Then the group of deck transforms acts simply transitively on the fiber $p^{-1}(x_0)$ and we have an isomorphism $\pi_1(Y, y_0) \cong \mathrm{Aut}(X \xrightarrow{p} Y)$.

A 28 Let (X, x_0) be a path-connected and locally path-connected space together with an action of a (discrete) group G. Assume that for every $x \in X$ there exists an open neighborhood $x \in U \subset X$ such that $U \cap gU = \emptyset$ for all $g \in G$, $g \neq e$.

(a) Prove that the quotient morphism $q : X \to X/G$ is a covering.

(b) Use exercise 27(d) to show that there is a surjective morphism of groups $\pi_1(X/G, q(x_0)) \to G$ with kernel isomorphic to $\pi_1(X, x_0)$.

Remark: For $X = \mathbb{R}$ and $G = \mathbb{Z}$, the last assertion shows again $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. In fact, the degree map $\deg : \pi_1(\mathbb{S}^1) \to \mathbb{Z}$ defined in the lecture is just an explicit description of the map $\pi_1(X/G) \to G$.

A 29 Consider the Klein Bottle $K = I \times I / \sim$, where $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, 1 - y)$.

(a) Compute $\pi_1(K)$ by finding a covering $\mathbb{R}^2 \to K$ and using one of the previous two exercises.

(b) Compute $\pi_1(K)$ again, but now using the theorem of Seifert-van Kampen.

Solutions to the exercises will be available from January 7, 2016 on, at
`https://www-m11.ma.tum.de/viehmann/topology/`