Technische Universität München
Zentrum Mathematik

WS 2015/16
Sheet 13

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Topology (MA 3241)

Exercises (to be handed in Thursday, 28.1.2016, before the lecture)

A 36 Let B and F be Hausdorff and locally compact topological spaces. Let $f : X \to B$ be a fiber bundle with typical fiber F and fix one atlas $A = \{U_i, g_i : f^{-1}(U_i) \to U_i \times F\}$ for f with transition functions $t_{i,j} = (g_j \circ g_i^{-1})^\sharp : U_i \cap U_j \to \mathrm{Aut}(F)$.

(a) Assume that $f : X \to B$ is the trivial fiber bundle. Prove that there are functions $\psi_i : U_i \to \mathrm{Aut}(F)$ on all U_i such that $t_{i,j} = \psi_j^{-1} \cdot \psi_i$ for all i, j.

(b) Prove the converse: Assume there are functions $\psi_i : U_i \to \mathrm{Aut}(F)$ on all U_i such that $t_{i,j} = \psi_j^{-1} \cdot \psi_i$ for all i, j, then $f : X \to B$ is the trivial fiber bundle.

A 37 In this exercise we construct lots of different $\mathbb{S}^1$-bundles over $\mathbb{S}^2$.

(a) Show that the Hopf fibration $h : \mathbb{S}^3 \to \mathbb{S}^2$ is not homeomorphic to the trivial $\mathbb{S}^1$-bundle over $\mathbb{S}^2$.

(b) Fix an integer $n \geq 1$ and a primitive root of unity $\zeta_n \in \mathbb{C}$. Define the nth lens space L_n as the quotient of $\mathbb{S}^3$ by the equivalence relation induced by $(z, z') \sim (\zeta_n z, \zeta_n z')$ for all (z, z'). Prove that the Hopf fibration induces a fibration $h_n : L_n \to \mathbb{S}^2$ with typical fiber $\mathbb{S}^1$.

(c) Show that no two fibrations $h_n : L_n \to \mathbb{S}^2$ are homeomorphic as $\mathbb{S}^1$-bundles over $\mathbb{S}^2$.

Remark: One can show that any $\mathbb{S}^1$-bundles over $\mathbb{S}^2$ is either homeomorphic to the trivial bundle or to one of the $h_n : L_n \to \mathbb{S}^2$ for some $n \geq 1$.

Solutions to the exercises will be available from January 28, 2016 on, at

`https://www-m11.ma.tum.de/viehmann/topology/`