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Topology (MA 3241)

Exercises (to be handed in Thursday, 19.11.2015, before the lecture)

A 13 In $\mathbb{R}^2$ (endowed with the euclidean topology) consider the following two subspaces:

(i) $X = \bigcup_{\substack{n\in\mathbb{Z}\\ n\geq 1}} \partial B_n((n,0))$ the union of all circles with center $(n,0)$ and radius n.

(ii) $Y = \bigcup_{\substack{n\in\mathbb{Z}\\ n\geq 1}} \partial B_{\frac{1}{n}}((\frac{1}{n},0))$ the union of all circles with center $(\frac{1}{n},0)$ and radius $\frac{1}{n}$.

Prove that X and Y are not homeomorphic, by showing that one is compact while the other is not.
Remark: Y is called the Hawaiian earring.

A 14 Let X be Hausdorff. Prove that any two compact subspaces A and B in X can be separated by open neighborhoods, i.e. X satisfies $(T4)$ for compact subsets.
Remark: In particular this exercise implies, that any compact Hausdorff space is $(T4)$.

A 15 Let X be Hausdorff and locally compact, i.e. X is Hausdorff and each point $x \in X$ admits a compact neighborhood. Then define its one-point compactification (or Alexandroff compactification) Y as the set $Y = X \cup \{\infty\}$ with the topology given by

$$\mathcal{T} = \{U \mid U \subset X \text{ open}\} \cup \{Y \setminus K \mid K \subset X \text{ compact}\}.$$

(a) Show that the canonical inclusion $i : X \hookrightarrow Y$ defines a homeomorphism between X and its image $i(X) \subset Y$, when $i(X)$ is endowed with the subspace topology.

(b) Show that Y is Hausdorff and compact.

(c) Let Y' be any other space together with an inclusion $i' : X \hookrightarrow Y'$ such that $Y' \setminus X$ consists of just one point. Show that if Y' has the properties of (a) and (b), then Y' is homeomorphic to Y.

(d) Show that the sphere $S^2 \subset \mathbb{R}^3$ is homeomorphic to the one-point compactification of $\mathbb{R}^2$.

Solutions to the exercises will be available from November 19, 2015 on, at

`https://www-m11.ma.tum.de/viehmann/topology/`