Technische Universität München
Zentrum Mathematik

WS 2015/16
Sheet 6

Prof. Dr. Eva Viehmann
M. Sc. Stephan Neupert

Topology (MA 3241)

Exercises (to be handed in Thursday, 26.11.2015, before the lecture)

A 16 We give an alternative proof of Tychonoff's theorem in this exercise, based on the **Lemma of Alexander:**
Let X be a topological space with a subbasis $\mathcal{S}$. Then X is compact, if every open cover $\{U_i\}_{i\in I} \subset \mathcal{S}$ of sets in the subbasis, contains a finite open subcover.

(a) Assume X is any topological space and $\{U_i\}_{i\in I}$ any cover of X. Then consider the set

$$\mathscr{C} = \big\{\{V_j\}_{j\in J} \,|\, \{U_i\}_{i\in I} \subset \{V_j\}_{j\in J} \text{ and } \{V_j\}_{j\in J} \text{ does not contain a finite subcover}\big\}$$

with the partial order given by inclusion. Assume $\mathscr{C}$ is nonempty. Prove that $\mathscr{C}$ contains a maximal element using the Lemma of Zorn.

(b) Fix now any subbasis $\mathcal{S}$ for X. Assume again that $\mathscr{C}$ is nonempty and let $\{V_j\}_{j\in J}$ be a maximal element in $\mathscr{C}$. Prove that for all $x \in X$ there exists some index $j \in J$ such that $x \in V_j \in \mathcal{S}$.

(c) Conclude that the Lemma of Alexander holds.

(d) Use the Lemma of Alexander to prove Tychonoff's theorem.

A 17 Our goal here is to prove the following statement:
Assume that X is Hausdorff, Y is locally compact and Z is any topological space. Then the adjunction map $\mathrm{Hom}(X\times Y, Z) \to \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$ *is a homeomorphism.*

(a) Assume that Y is locally compact and X, Z are any topological spaces. Let $C_X \subset X$ and $C_Y \subset Y$ be compact subspaces, $U \subset Z$ open. Prove that the adjunction map defines a bijection between $\mathrm{M}(C_X \times C_Y, U) \subset \mathrm{Hom}(X\times Y, Z)$ and $\mathrm{M}(C_X, \mathrm{M}(C_Y, U)) \subset \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$.

(b) Assume X is Hausdorff, T any topological space with a basis $\mathcal{B}$ of the topology. Prove that $\{\mathrm{M}(C_X, U) \,|\, C_X \subset X \text{ compact}, U \in \mathcal{B}\}$ is a subbasis for the CO-topology on $\mathrm{Hom}(X, T)$.

(c) Prove that under the same assumptions as in part (b), $\{\mathrm{M}(C_X, U) \,|\, C_X \subset X \text{ compact}, U \in \mathcal{S}\}$ is a subbasis for the CO-topology on $\mathrm{Hom}(X, T)$, whenever $\mathcal{S}$ is a subbasis for the topology on T.

(d) Conclude that the statement at the beginning holds, i.e. that the adjunction map is a homeomorphism.

Remark: You may use the following lemma (known from the exercise classes) without proof:
If C is compact and Hausdorff, $x \in C$ any point and $U \subset C$ an open neighborhood of x, then there exists a compact neighborhood $A \subset C$ of x with $A \subset U$.

Solutions to the exercises will be available from November 26, 2015 on, at
`https://www-m11.ma.tum.de/viehmann/topology/`