Some additional exercises...

Let k be a field and assume (for simplicity) that k is algebraically closed. Our aim is to investigate the scheme

$$X = \text{Proj}\,(k[s,t,x,y]/(sx-ty))$$

where $k[s,t,x,y]/(sx-ty)$ is the graded ring with grading given by $\deg(s) = \deg(t) = 0$ and $\deg(x) = \deg(y) = 1$.

Question 1: Show that $k[s,t,x,y]/(sx-ty)$ is indeed a graded ring. Describe all elements of degree 2.

Question 2: Compute $\mathcal{O}_X(X)$, $\mathcal{O}_X(D(s))$ and $\mathcal{O}_X(D(x))$.

Question 3: Show that $D(x) \cong \mathbb{A}^2_k \cong D(y)$ are affine open subschemes of X. Show that they cover X.

Question 4: Is X reduced? Is it irreducible?

Question 5: Show that there is a unique morphism of schemes $\pi : X \to \mathbb{A}^2_k = \text{Spec}(k[\overline{s},\overline{t}])$ such that the map $\pi^\sharp : \mathcal{O}_{\mathbb{A}^2_k}(\mathbb{A}^2_k) \to \mathcal{O}_X(X)$ on global sections satisfies $\pi^\sharp(\overline{s}) = s$ and $\pi^\sharp(\overline{t}) = t$.

Question 6: Let $p \in \mathbb{A}^2_k$ be a closed point. Describe the fiber $\pi^{-1}(p)$ (as a scheme). (Hint: Recall that if $S = \oplus_i S_i$ is graded and $S_0 \to R_0$ is any morphism of rings, then there is an isomorphism $\text{Proj}(S) \times_{\text{Spec}\, S_0} \text{Spec}\, R_0 \cong \text{Proj}(S \otimes_{S_0} R_0)$, where we use the grading $S \otimes_{S_0} R_0 = \oplus_i (S_i \otimes_{S_0} R_0)$.)

Question 7: Is π quasi-compact, of finite type, quasi-finite, finite, affine, separated, proper? (Hint: Feel free to embed X into a projective space or to base-change X to some closed point of $\mathbb{A}^2_k$.)

Question 8: Use the fiber over the point $(\overline{s},\overline{t}) \in \mathbb{A}^2_k$ to show that $D(s+1) \subset X$ is not affine.

Question 9: Compute the evaluation of the sheaves $\Omega_{X/k}$ and $\Omega_{X/\mathbb{A}^2_k}$ on the open affine subscheme $D(x) \subset X$.

Question 10: Consider $\mathbb{P}^1_k = \text{Proj}\, k[\overline{x},\overline{y}]$ (where $\deg(\overline{x}) = \deg(\overline{y}) = 1$). Show that

$$pr^\sharp : k[\overline{x},\overline{y}] \to k[s,t,x,y]/(sx-ty) \qquad \overline{x} \mapsto x \ , \ \overline{y} \mapsto y$$

is a morphism of graded rings. Deduce that it defines a morphism $pr : X \to \mathbb{P}^1_k$.

Question 11: Show that there is a cover of $\mathbb{P}^1_k$ by open subschemes U_i such that there are isomorphisms $pr^{-1}(U_i) \cong U_i \times_{\text{Spec}\, k} \mathbb{A}^1_k$ for each U_i.

Question 12: Is pr quasi-compact, of finite type, quasi-finite, finite, affine, separated, proper?

Question 13: Prove that X is not isomorphic to $\mathbb{P}^1_k \times_{\text{Spec}\, k} \mathbb{A}^1_k$. (Hint: Compare global sections.)

Question 14*: Consider the invertible sheaf $\mathcal{O}_{\mathbb{P}^1_k}(1)$ on $\mathbb{P}^1_k$. Over every open affine $U \subset \mathbb{P}^1_k$ take the symmetric algebra $Sym(\mathcal{O}_{\mathbb{P}^1_k}(1)(U))$ as an algebra over $\mathcal{O}_{\mathbb{P}^1_k}(U)$. Thus we may cosider the schemes $\text{Spec}\left(Sym(\mathcal{O}_{\mathbb{P}^1_k}(1)(U))\right) \to U$. Show that for any two such open subsets $U, V \subset \mathbb{P}^1_k$ the canonical isomorphism $(\mathcal{O}_{\mathbb{P}^1_k}(1)|_U)(U \cap V) \cong (\mathcal{O}_{\mathbb{P}^1_k}(1)|_V)(U \cap V)$ induces an isomorphism

$$\text{Spec}\left(Sym(\mathcal{O}_{\mathbb{P}^1_k}(1)(U))\right) \times_U (U \cap V) \cong \text{Spec}\left(Sym(\mathcal{O}_{\mathbb{P}^1_k}(1)(V))\right) \times_V (U \cap V)$$

over $U \cap V$. Use these isomorphisms to glue the schemes $\text{Spec}\left(Sym(\mathcal{O}_{\mathbb{P}^1_k}(1)(U))\right)$ (for all open affines U) to a scheme $\underline{\text{Spec}}_{\mathbb{P}^1_k}\left(Sym(\mathcal{O}_{\mathbb{P}^1_k}(1))\right)$. Finally construct an isomorphism

$$X \cong \underline{\text{Spec}}_{\mathbb{P}^1_k}\left(Sym(\mathcal{O}_{\mathbb{P}^1_k}(1))\right)$$

of schemes over $\mathbb{P}^1_k$.