Technische Universität München
Zentrum Mathematik

WS 2018/19
Sheet 1

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Algebraic Geometry

To be handed in October 24, before the seminar talk.

We fix an algebraically closed field k in all the exercises.

Exercise 1. Let X be the union of the n coordinate axes in $\mathbb{A}^n(k)$. Prove that X is an affine variety and determine the corresponding radical ideal in $k[x_1, ..., x_n]$.

Exercise 2. Prove that the Zariski topology on $\mathbb{A}^2(k)$ is different from the product topology on $\mathbb{A}^1(k) \times \mathbb{A}^1(k)$. (Recall that for two topological spaces X and Y, the product topology on $X \times Y$ is the topology generated by all the subsets $U \times V$, where $U \subset X$ and $V \subset Y$ are open subsets.)

Exercise 3. Recall that a topological space is called *irreducible* if it cannot be written as a union of two proper closed subsets.

1. Suppose X is an irreducible topological space. Show that every non-empty open subset of X is dense.
2. Suppose X is a topological space, and $Y \subset X$ a subset. Show that the closure $\overline{Y}$ of Y in X is irreducible if and only if Y is irreducible.

Exercise 4. Let X be the affine variety in $\mathbb{A}^3(k)$ defined by two equations $x(z-1) = 0$ and $x^2 = yz$. Give an explicit decomposition $X = X_1 \cup X_2 \cup ... \cup X_r$, such that the X_i are irreducible closed subsets of X for all i, and such that for $i \neq j$ one has $X_i \not\subseteq X_j$. Prove that this decomposition is unique. The X_i are called the *irreducible components* of X.

Remark to exercise 3 Here is an example that will appear later in the course. Suppose we have a scheme X and a point $x \in X$. Then the closure Z of the set $\{x\}$ is an irreducible closed sub-scheme of X. It can happen that Z is much bigger than $\{x\}$, even equal to the whole scheme X. In the latter case we call x the *generic point* of Z.
Remark to exercise 4: A topological space X is called *Noetherian* if every chain of closed subsets

$$Z_1 \supset Z_2 \supset \cdots$$

is stationary. Then one can prove that every Noetherian topological space has a unique irreducible component decomposition, and that the underlying topological space of an affine variety is Noetherian.

References. We will need some basic notions of topology: topological spaces, basis of a topology, continuous functions, product topology. You can find them in almost every textbook on topology, for example:
Bredon “Algebraic topology”,
Laures/Szymik “Grundkurs Topologie”.

In case of questions please send us an email or contact us before or after the seminar/problem session.
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