Technische Universität München
Zentrum Mathematik

WS 2018/19
Sheet 13

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Algebraic Geometry

To be handed in January 30, before the lecture.

Exercise 1. Let k be a field and $G = \mathrm{GL}_{n,k} \subseteq \mathbb{A}_k^{n^2}$ the open subscheme $D(\det)$. Let e be the unit matrix.

1. Show that we can identify T_eG with k^{n^2}, viewed as the set of $n \times n$-matrices over k.
2. Let $m : G \times G \to G$ be the multiplication map. Show that $dm_{(e,e)} : T_eG \times T_eG \to T_eG$ is given by $(v, w) \mapsto v + w$.
3. Show that $d\det_e : k^{n^2} \to k$ induced by $\det : G \to \mathrm{GL}_{1,k}$ is the trace map.

Exercise 2. Let R be an algebra over a field k and let M be an R-module. Then a map $\delta : R \to M$ is a derivation if δ is additive and for all $x, y \in R$ we have

$$\delta(xy) = x\delta(y) + y\delta(x).$$

Let X be a k-scheme and $x \in X$ a closed point with $\kappa(x) = k$. View $\kappa(x)$ as an $\mathcal{O}_{X,x}$-module via the projection $\mathcal{O}_{X,x} \to \kappa(x)$. Show that there is a canonical isomorphism

$$\{ \text{ derivations } \delta : \mathcal{O}_{X,x} \to \kappa(x)\} \cong T_xX.$$

Exercise 3. Let K/k be a finite extension of fields. Let $X = \mathrm{Spec}K$ and consider X as a k-scheme. Let $x : \mathrm{Spec}K \to X$ be the identity map. Show that $T_xX = 0$, but $T_x(X/k) = 0$ if and only if K/k is separable.

Exercise 4. Let k_0 be a field of characteristic $p > 0$ and let $k = k_0(t)$. Let $X = V(y^2 - x^p + t) \subseteq \mathbb{A}_k^2 = \mathrm{Spec}k[x, y]$. Show that every local ring of X is a regular local ring, but that X is not smooth over k.

In case of questions please send us an email or contact us before or after the seminar/problem session.
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